# Computational Astrophysics

## Lecture 3: Magnetic fields

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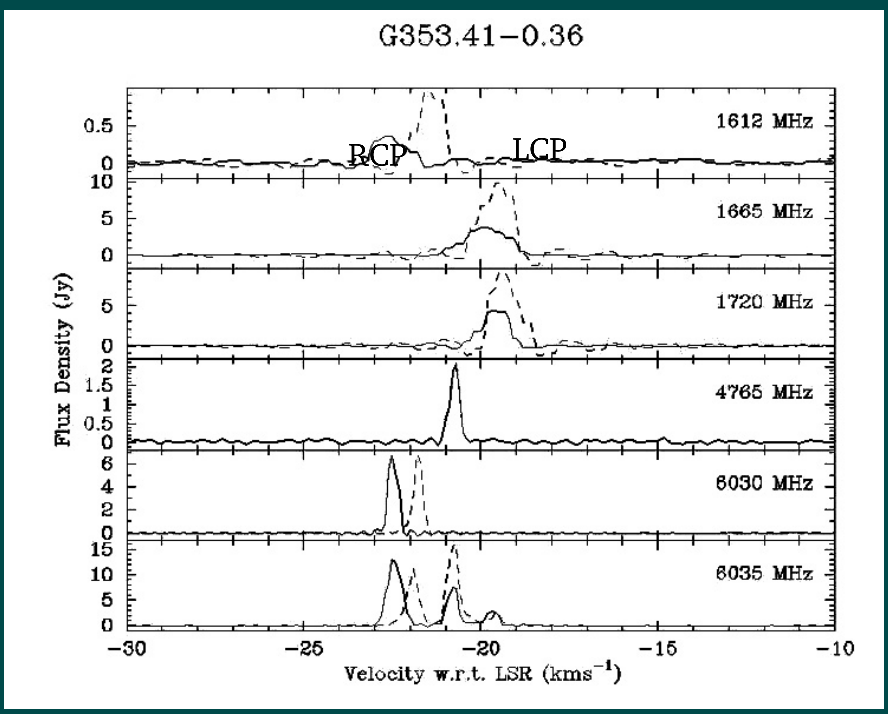
# Magnetic fields in astrophysics

Because positive and negative charges can rearrange themselves in response to electric fields, on large scales we typically have **E** = 0.

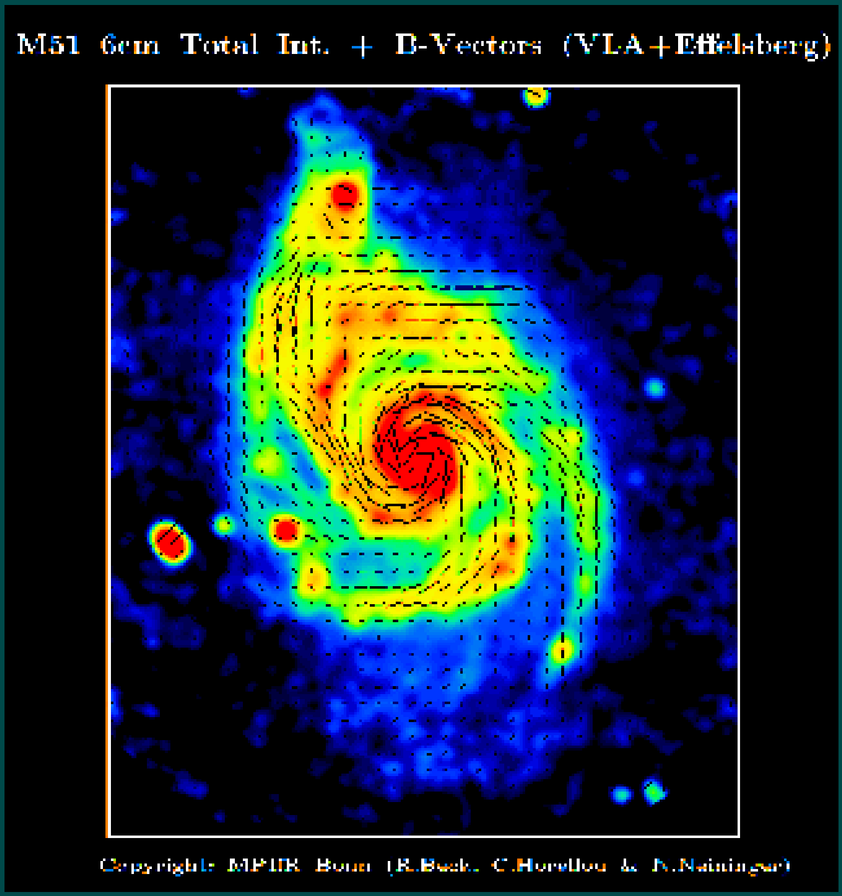
However, no magnetic charges, so nonzero **B** tends to persist.

Observational indications of **B** fields:

- Faraday rotation
- Synchrotron emission
- Zeeman splitting




Zeeman splitting for different OH masers detected toward G353.41-0.36 (Ellingsen et al. 2002)




**B** field from Faraday rotation in M51 (MPIfR)

# Plasmas and the MHD limit

We are implicitly assuming that the *plasma parameter* is large for all species:

e.g., for electrons $$\Lambda \equiv n_e \lambda_{D,e}^3 = \left(\frac{k_B}{4\pi e^2}\right)^{3/2} n_e^{-1/2} T_e^{3/2} \gg 1$$

Recall that this assumption was necessary for Debye shielding to work. If it does not work, we may have large-scale charge separation, in which case we will be unable to work in the *single-fluid MHD limit*.

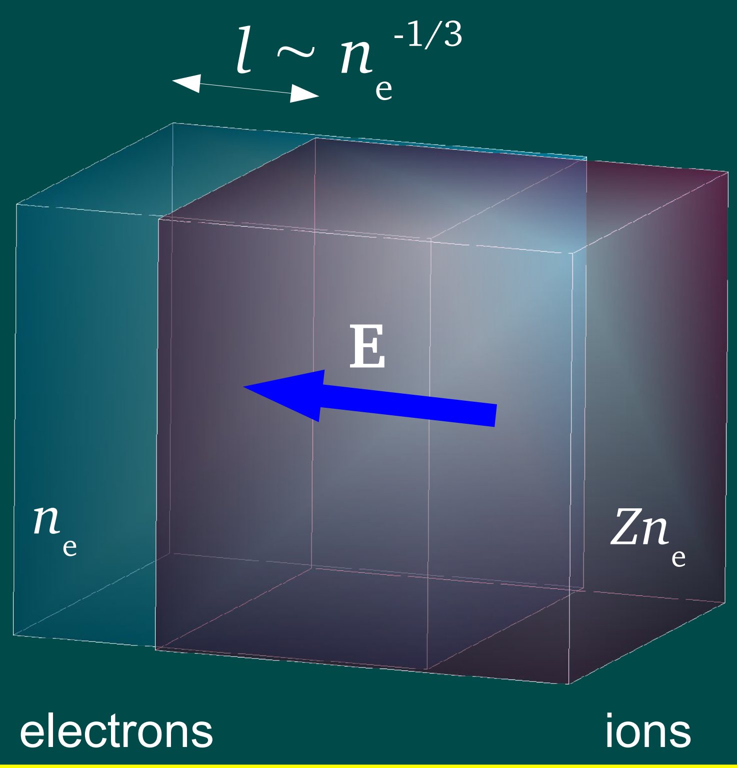
Only a small amount of charge separation is needed to produce a huge restoring electric field:

$$E \sim \Sigma \sim n_e l e$$

$$f \sim \Sigma E \sim n_e l e E \sim n_e^2 l^2 e^2$$

$$a \sim \frac{f}{n_e l m_e} \sim \frac{n_e l e^2}{m_e}$$

$$a \sim \frac{l}{\tau^2} \quad \Rightarrow \quad \tau \sim \frac{m_e^{1/2}}{n_e^{1/2} e}$$




For $n_e = 1\ \mathrm{cm}^{-3}$, $\tau \sim 7\times10^{-5}$ sec !

# Conduction in an electrically neutral plasma

If we ignore displacement currents (so material currents are the only source of **B**),

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}_c$$

Electric current $\mathbf{j}_\mathrm{c}$ comes from differential motion of + and – charges (*drift*). How big are typical drift velocities $\mathbf{v}_\mathrm{e}$?

Overall charge neutrality: $$\rho_\mathrm{c} \equiv Zen_\mathrm{i} - en_\mathrm{e} = 0$$

Electric current due to differing mean velocities of ions ($\mathbf{u}_\mathrm{i}$) and electrons ($\mathbf{u}_\mathrm{e}$):

$$\mathbf{j}_c = Zen_\mathrm{i}\,\mathbf{u}_\mathrm{i} - e\,n_\mathrm{e}\,\mathbf{u}_\mathrm{e} = -e\,n_\mathrm{e}\,\mathbf{v}'_\mathrm{e}$$

where the electron drift velocity relative to ions is

$$\mathbf{v}'_\mathrm{e} \equiv \mathbf{u}_\mathrm{e} - \mathbf{u}_\mathrm{i}$$

*Example*: solar magnetic fields

$B \sim 10^3$ G convection zone depth $L \sim 2\times10^{10}$ cm density $n_\mathrm{e} \sim 10^{23}$ cm$^{-3}$

$$\Rightarrow v_\mathrm{e} \sim \frac{cB}{4\pi\,e\,n_\mathrm{e}\,L} \sim 10^{-12}\ \mathrm{cm\ s}^{-1}$$

Under most circumstances the velocity difference between ions and electrons can be ignored.

# Conduction in an electrically neutral plasma – 2

In ion rest frame (primed), the electron equation of motion is

$$\cancel{m_e \frac{d\mathbf{v}'_e}{dt}} = -e\left(\mathbf{E}' + \cancel{\frac{\mathbf{v}'_e}{c}\times\mathbf{B}'}\right) - m_e \nu_c \mathbf{v}'_e \cancel{- m_e \nu_e^{(n)}(\mathbf{u}_e - \mathbf{u}_n)} \cancel{+ \text{inertial terms}}$$

where $\nu_c$ is the mean electron-ion collision frequency.

We ignore:

- the drag effect on the electrons due to collisions with neutral species ($\times$)
- inertia of the electrons ($\times$)
- gyroscopic motion of electrons ($\times$)

Latter two conditions require relatively weak magnetic field gradients.

Thus the terminal (drift) velocity is $\mathbf{v}'_e = -\dfrac{e}{m_e \nu_c}\mathbf{E}'$

and the corresponding electrical current is $\mathbf{j}_c = \mathbf{j}'_c = -e\, n_e \mathbf{v}'_e = \sigma \mathbf{E}'$

where $\sigma$ is the electrical conductivity: $\sigma = \dfrac{n_e e^2}{m_e \nu_c}$

In the lab frame we have $(u_i/c \ll 1)$ $\quad \mathbf{B}' = \mathbf{B} \qquad \mathbf{E}' = \mathbf{E} + \dfrac{\mathbf{u}_i}{c}\times\mathbf{B}$

# Equations of magnetohydrodynamics

General idea: solve coupled Maxwell equations and hydrodynamic equations

We want a reduced form of Maxwell's equations that:

- Does not explicitly include electric field $\mathbf{E}$;
- Does not include displacement current (source of rapidly varying radiation fields).

$$\nabla \cdot \mathbf{E} = 4\pi \rho_c$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}_c + \cancel{\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}}$$

Electric field in lab frame due to electron-ion drift (nonrelativistic):

$$\mathbf{E} = \frac{1}{\sigma}\mathbf{j}_c - \frac{\mathbf{u}_i}{c}\times\mathbf{B} = \frac{c}{4\pi\sigma}\nabla\times\mathbf{B} - \frac{\mathbf{u}_i}{c}\times\mathbf{B}$$

Substitute this expression into Faraday's law to obtain:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\times(\mathbf{B}\times\mathbf{u}_i) = -\nabla\times\left(\frac{c^2}{4\pi\sigma}\nabla\times\mathbf{B}\right)$$

# Equations of magnetohydrodynamics – 2

Define the *electrical resistivity* via

$$\eta \equiv \frac{c^2}{4\pi\sigma} = \frac{c^2 m_{\mathrm{e}} \nu_{\mathrm{c}}}{4\pi n_{\mathrm{e}} e^2} = \frac{c^2 \nu_{\mathrm{c}}}{\omega_{\mathrm{pe}}^2}$$

where $\omega_{\mathrm{pe}}$ is the electron plasma frequency. Then, taking $\mathbf{u}_{\mathrm{i}} \approx \mathbf{u}$, we have

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{u}) = -\nabla \times (\eta \nabla \times \mathbf{B})$$

$$\nabla \cdot \mathbf{B} = 0$$

Because we assume overall charge neutrality, the Lorentz force on the matter per unit volume is

$$\mathbf{f}_{\mathrm{L}} = Zen_{\mathrm{i}}\left(\mathbf{E} + \frac{\mathbf{u}_{\mathrm{i}}}{c} \times \mathbf{B}\right) - en_{\mathrm{e}}\left(\mathbf{E} + \frac{\mathbf{u}_{\mathrm{e}}}{c} \times \mathbf{B}\right)$$

$$= \frac{1}{c}\mathbf{j}_{\mathrm{c}} \times \mathbf{B}$$

$$= \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

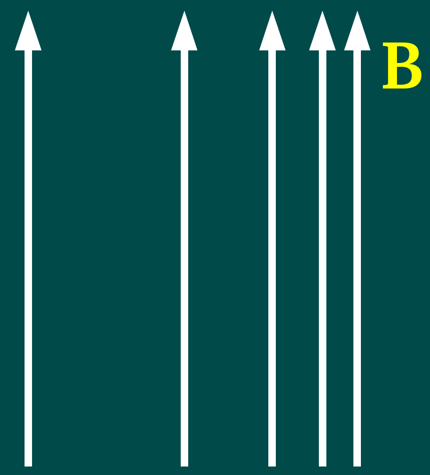
$$= \frac{1}{4\pi}(\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{8\pi}\nabla\left(|\mathbf{B}|^2\right)$$

# Equations of magnetohydrodynamics – 3

The two terms in the Lorentz force expression represent magnetic tension and magnetic pressure:

$$\mathbf{f}_{\mathrm{L}} = \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{8\pi}\nabla\left(|\mathbf{B}|^2\right)$$


Only magnetic tension




Only magnetic pressure

Can write the Lorentz force as $\mathbf{f}_{\mathrm{L}} = \nabla\cdot\boldsymbol{T}$

where the Maxwell stress tensor is

$$\boldsymbol{T} \equiv \frac{\mathbf{B}\mathbf{B}}{4\pi} - \frac{|\mathbf{B}|^2}{8\pi}\boldsymbol{I}$$

# Equations of magnetohydrodynamics – 4

We also need to consider the heating effect of Ohmic dissipation:

$$\text{Heating rate per unit volume} = \mathbf{j}_c \cdot \mathbf{E} = \frac{1}{\sigma} |\mathbf{j}_c|^2 = \frac{\eta}{4\pi} |\nabla \times \mathbf{B}|^2$$

Thus the full set of MHD equations (lab frame, $u/c \ll 1$, charge neutrality, electrons and ions treated as single fluid):

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{u}) = -\nabla \times (\eta \nabla \times \mathbf{B})$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla P + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot \left[ (\rho E + P) \mathbf{u} \right] = \frac{\eta}{4\pi} |\nabla \times \mathbf{B}|^2$$

# Force-free B fields and plasma $\beta$

If we have a *force-free magnetic field*, then

$$4\pi\,\mathbf{f}_{\mathrm{L}} = (\mathbf{B}\cdot\nabla)\,\mathbf{B} - \frac{1}{2}\nabla\left(|\mathbf{B}|^2\right) = 0$$

We compare the importance of thermal pressure and magnetic pressure using the *plasma beta parameter*:

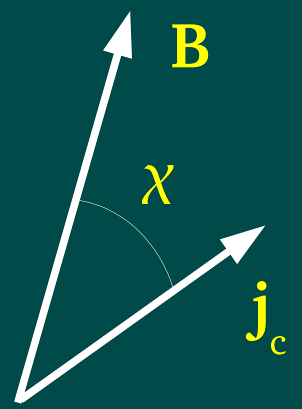
$$\beta \equiv \frac{8\pi P}{B^2}$$

For low-$\beta$ plasmas it is often a good approximation to assume the magnetic field to be force-free:

$$\rho\frac{D\mathbf{u}}{Dt} = \frac{1}{c}\mathbf{j}_{\mathrm{c}}\times\mathbf{B} - \nabla P = 0$$

$$\sin\chi = \frac{|\mathbf{j}_{\mathrm{c}}\times\mathbf{B}|}{j_{\mathrm{c}}B} = \frac{4\pi|\nabla P|}{|\nabla\times\mathbf{B}|B}$$

$$\sin\chi \sim \frac{4\pi P L_{\mathrm{M}}}{B^2 L_{\mathrm{P}}} \sim \frac{1}{2}\beta\frac{L_{\mathrm{M}}}{L_{\mathrm{P}}}$$



# Flux-freezing

The $\mathbf{B}$-field equation looks like the vorticity equation:

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times(\mathbf{B}\times\mathbf{u})=-\nabla\times(\eta\nabla\times\mathbf{B}) \qquad \frac{\partial \omega}{\partial t}+\nabla\times(\omega\times\mathbf{u})=0$$

In the absence of resistive effects, $\mathbf{B}$ is conserved like vorticity... just as the vorticity equation implies conservation of circulation, the *ideal $\mathbf{B}$-field equation* implies conservation of *magnetic flux*:

$$\Phi\equiv\int_A \mathbf{B}\cdot\mathbf{n}\,dA$$

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times(\mathbf{B}\times\mathbf{u})=0 \quad\rightarrow\quad \frac{D\Phi}{Dt}=0$$

The resistivity term behaves like a diffusive dissipation term:

$$\begin{aligned}\nabla\times(\eta\nabla\times\mathbf{B})&=(\nabla\eta)\times(\nabla\times\mathbf{B})+\eta\nabla\times\nabla\times\mathbf{B}\\&=(\nabla\eta)\times(\nabla\times\mathbf{B})+\eta\nabla(\nabla\cdot\mathbf{B})-\eta\nabla^2\mathbf{B}\\&=(\nabla\eta)\times(\nabla\times\mathbf{B})-\eta\nabla^2\mathbf{B}\end{aligned}$$

If resistivity is constant we have

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times(\mathbf{B}\times\mathbf{u})=\eta\nabla^2\mathbf{B}$$

# Characteristics of the MHD equations

Consider adiabatic, linear perturbations about a static, homogeneous state:

$$\rho=\rho_0(1+\delta) \quad \mathbf{u}=\mathbf{u} \quad P=P_0(1+\gamma\,\delta) \quad \mathbf{B}=B_0(\hat{\mathbf{n}}+\mathbf{b})$$

Substituting into the single-fluid MHD equations, we obtain

$$\frac{\partial\delta}{\partial t}+\nabla\cdot\mathbf{u}=0$$

$$\rho_0\frac{\partial\mathbf{u}}{\partial t}=-P_0\gamma\nabla\delta+\frac{B_0^2}{4\pi}(\nabla\times\mathbf{b})\times\hat{\mathbf{n}}$$

$$\frac{\partial\mathbf{b}}{\partial t}+\nabla\times(\hat{\mathbf{n}}\times\mathbf{u})=0$$

Assuming a solution of the form $e^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{x})}$ gives us

$$i\omega\delta-i\mathbf{k}\cdot\mathbf{u}=0$$

$$i\omega\mathbf{u}=\frac{\gamma P_0}{\rho_0}i\mathbf{k}\delta-\frac{B_0^2}{4\pi\rho_0}(i\mathbf{k}\times\mathbf{b})\times\hat{\mathbf{n}}$$

$$i\omega\mathbf{b}-i\mathbf{k}\times(\hat{\mathbf{n}}\times\mathbf{u})=0$$

# Characteristics of the MHD equations – 2

Define

sound speed $a^2 = \dfrac{\gamma P_0}{\rho_0}$ Alfvén speed $v_A^2 = \dfrac{B_0^2}{4\pi\rho_0}$

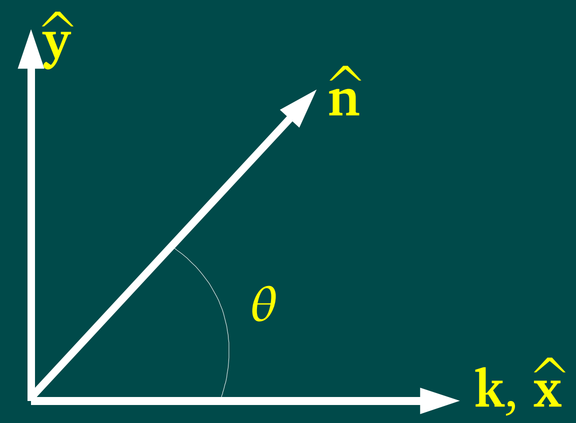
and use first and third equations to eliminate $\delta$ and $\mathbf{b}$:

$$\omega^2 \mathbf{u} + v_A^2 [\mathbf{k}\times[\mathbf{k}\times(\hat{\mathbf{n}}\times\mathbf{u})]]\times\hat{\mathbf{n}} - a^2 \mathbf{k}\,\mathbf{k}\cdot\mathbf{u} = 0$$

If we take $\mathbf{k}$ along $x$-axis and $\mathbf{n}$ in $xy$-plane, with $\theta$ the angle between $\mathbf{k}$ and $\mathbf{n}$, and expand the vector cross products, we obtain

$$\begin{aligned}&[\omega^2 - k^2 v_A^2 \cos^2\theta]\hat{\mathbf{u}} + \\ &[-k^2(v_A^2 + a^2)u_x + k^2 v_A^2 \cos\theta(u_x\cos\theta + u_y\sin\theta) + k^2 v_A^2 \cos^2\theta\, u_x]\hat{\mathbf{x}} + \\ &[k^2 v_A^2 \sin\theta\cos\theta\, u_x]\hat{\mathbf{y}} = 0\end{aligned}$$

This equation allows modes with $u_z = 0$ and modes with $u_z \neq 0$.



# Characteristics of the MHD equations – 3

Three characteristic families (in addition to usual entropy waves at velocity $\mathbf{u}$):

(1) **Alfvén waves** $u_x = u_y = 0, u_z \neq 0$

Disturbance travels along field line with speed $v_A \cos\theta$: $\omega^2 - k^2 v_A^2 \cos^2\theta = 0$

These are transverse magnetic tension waves.

(2) **Fast MHD waves** $u_x, u_y \neq 0; u_z = 0$

Using $\mathbf{u} = (u_x, u_y, 0)$ in the characteristic equation we obtain a matrix equation with right-hand side 0; for a nontrivial solution to exist the determinant must be zero:

$$\omega^4 - k^2 (v_A^2 + a^2)\omega^2 + k^4 v_A^2 a^2 \cos^2\theta = 0$$

Solution is $\omega^2 = \frac{1}{2}\left[(v_A^2 + a^2) \pm \left[(v_A^2 + a^2)^2 - 4 v_A^2 a^2 \cos^2\theta\right]^{1/2}\right] k^2$

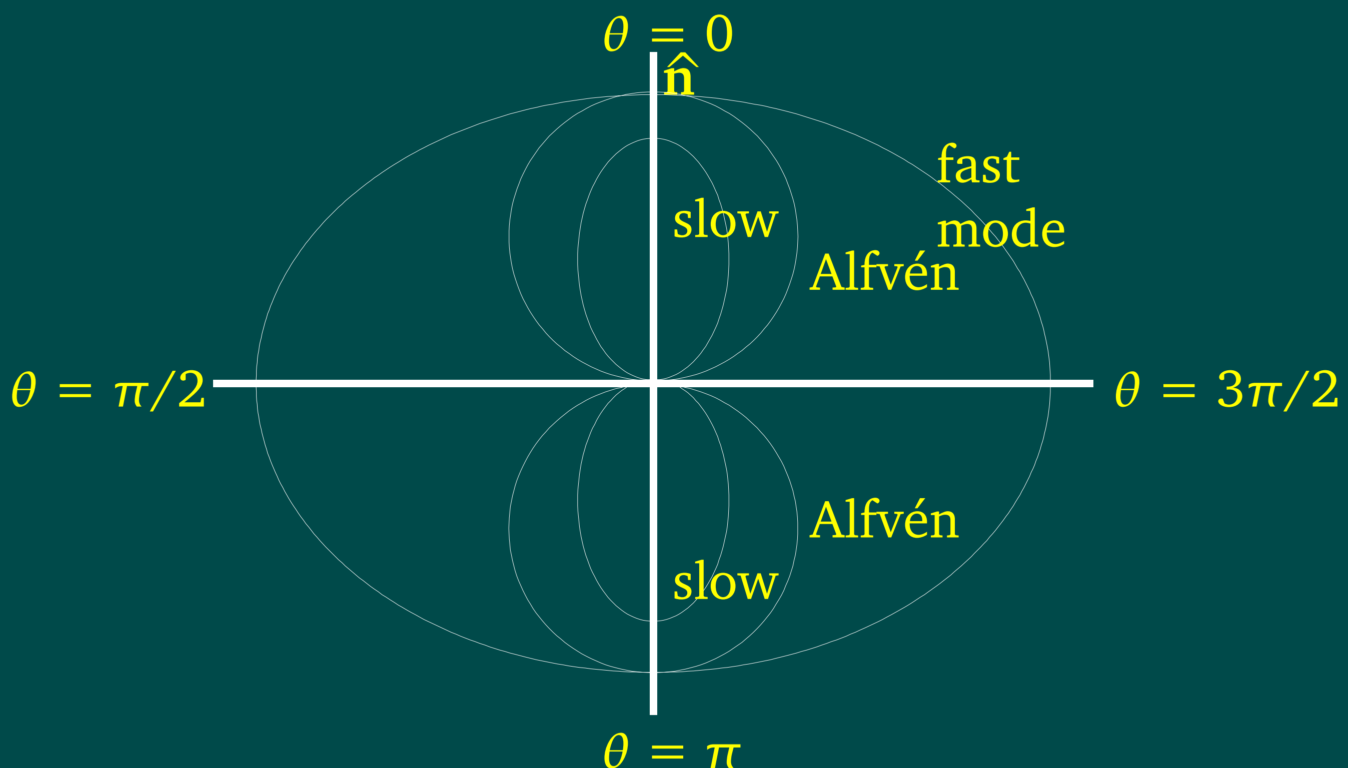
The fast waves correspond to the + sign in this MHD wave dispersion relation. When $\cos\theta = 0$ they are purely compressional *magnetosonic* waves.

(3) **Slow MHD waves** $u_x, u_y \neq 0; u_z = 0$

Correspond to the – sign in the MHD wave dispersion relation.

# Characteristics of the MHD equations – 4

Character of the magnetosonic waves depends on angle $\theta$: (e.g., for $v_A > a$)




For $\mathbf{k} \parallel \hat{\mathbf{n}}$, we have an acoustic wave and an Alfvén wave;
For $\mathbf{k} \perp \hat{\mathbf{n}}$, we have a (static) slow magnetosonic wave and a fast magnetosonic wave;
For other angles, we have a combination of transverse and compressional waves.

Notice that wave speeds are degenerate in certain directions.

# Computational magnetohydrodynamics

Numerical MHD differs from numerical hydrodynamics in several significant ways:

- There is an additional equation to solve (the induction equation for $\mathbf{B}$)
- There are more families of characteristics to consider
- The equations are not always strictly hyperbolic, or even hyperbolic
- The solution must maintain the divergence-free character of $\mathbf{B}$ everywhere

These factors combine to make numerical MHD significantly more difficult than plain hydro (though perhaps not as difficult as radiation hydro!)

# Ensuring div B = 0

What happens when we allow $\nabla\cdot\mathbf{B}\neq 0$?

Recall that the Lorentz force per unit volume is

$$\mathbf{f}_{\mathrm{L}}=\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}-\frac{1}{8\pi}\nabla\left(|\mathbf{B}|^2\right)$$

We wrote this as a conserved flux of a stress tensor,

$$\mathbf{f}_{\mathrm{L}}=\nabla\cdot\boldsymbol{T}\,,\quad \boldsymbol{T}\equiv\frac{\mathbf{B}\mathbf{B}}{4\pi}-\frac{|\mathbf{B}|^2}{8\pi}\boldsymbol{I}$$

But doing this required that we have $\nabla\cdot\mathbf{B}=0$:

$$\nabla\cdot(\mathbf{B}\,\mathbf{B})=\mathbf{B}\,\nabla\cdot\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{B}$$

If $\nabla\cdot\mathbf{B}\neq 0$, we effectively have a magnetic force component parallel to $\mathbf{B}$.

In addition to allowing unphysical magnetic acceleration along the field lines, this also breaks energy conservation.

# Ensuring $\operatorname{div} \mathrm{B} = 0$

A solution to the continuum equations that initially satisfies the divergence constraint will do so for all subsequent time:

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$$

However, a solution to the difference equations that initially satisfies the discretized divergence constraint is *not* guaranteed to do so for all subsequent time.

So what can we do?

- Ignore the problem. (**bad**)
- Carry around the vector potential $\mathbf{A}$, and compute $\mathbf{B}$ from it: $\mathbf{B} = \nabla \times \mathbf{A}$. Then $\nabla \cdot \mathbf{B} = 0$ to the extent that our difference approximation to $\nabla \cdot \nabla \times$ always gives zero.
- Build the divergence constraint into our differencing scheme.
- Apply artificial diffusion designed to *damp* the component of $\mathbf{B}$ with nonzero divergence.
- Every few timesteps, *project out* the component of $\mathbf{B}$ with nonzero divergence.

# Constrained transport method

(Evans & Hawley 1988)

In this finite-difference method, we use a staggered mesh and carry the magnetic flux $\Phi$ rather than the magnetic field. In ideal MHD, the flux through a surface $S(t)$ evolves via

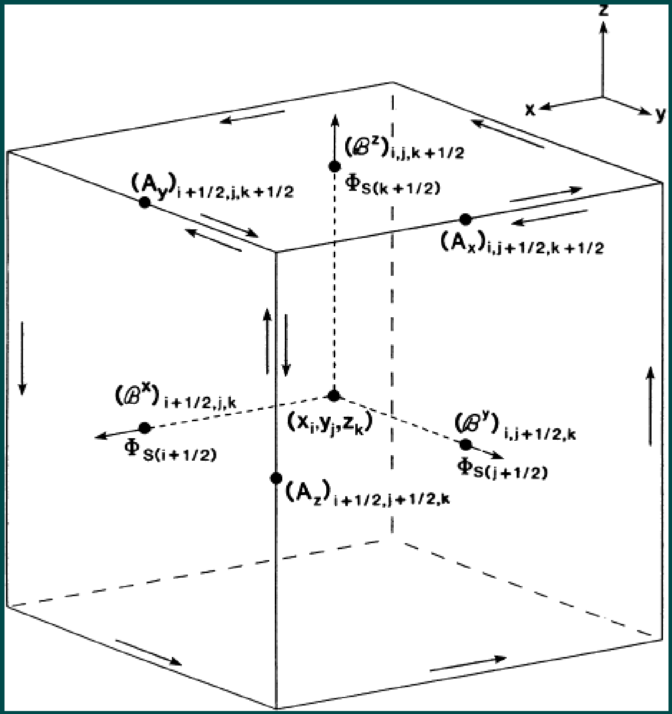
$$\frac{d}{dt}\Phi_{S(t)} = \frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot\mathbf{n}\,dS$$

$$= \int_{S(t)} \nabla\times(\mathbf{u}\times\mathbf{B})\cdot\mathbf{n}\,dS$$

$$= \oint_{\partial S(t)} \mathbf{u}\times\mathbf{B}\cdot d\,\mathbf{l}$$

$$= -\mathscr{E}$$

where $\mathscr{E}$ is the electromotive force (EMF).




We can take $S$ to be each face of a zone. The EMF is then a line integral over the edges of the zone. By storing fluxes at the centers of cell faces, we can "conservatively" compute the EMF. The field components are, e.g.,

$$(B_x)_{i+1/2,jk} = \frac{1}{\Delta y_j \Delta z_k}\Phi_{S_x(i+1/2)}$$

# Constrained transport method – 2

The div B constraint is equivalent to the requirement that the net flux through the zone be zero:

$$\Phi_{S_x(i+1/2)} - \Phi_{S_x(i-1/2)} + \Phi_{S_y(j+1/2)} - \Phi_{S_y(j-1/2)} + \Phi_{S_z(k+1/2)} - \Phi_{S_z(k-1/2)} = 0$$

If we define field components also at the face centers, this is equivalent to

$$\frac{1}{\Delta x_i}\left[(B_x)_{i+1/2,jk} - (B_x)_{i-1/2,jk}\right] + \frac{1}{\Delta y_j}\left[(B_y)_{i,j+1/2,k} - (B_y)_{i,j-1/2,k}\right] +$$

$$\frac{1}{\Delta z_k}\left[(B_z)_{ij,k+1/2} - (B_z)_{ij,k-1/2}\right] = 0$$

which is the difference version of $\nabla \cdot \mathbf{B} = 0$.

Each edge segment contributes to the EMF (when traversed in a right-handed fashion) a quantity $F$. For example, the segment along $x = x_{i+1/2}$ and $y = y_{j+1/2}$ with length $\Delta z_k$ contributes

$$F_{i+1/2,j+1/2,k} \equiv -\int_{z_{k-1/2}}^{z_{k+1/2}} (u_x B_y - u_y B_x)\, dz$$

to $d\Phi_{S_x(i+1/2)}/dt$ . The same amount is subtracted from the expression for $d\Phi_{S_y(j+1/2)}/dt$ .

# Constrained transport method – 3

Thus we have the following update method for fluxes:

$$\Phi^{n+1}_{S_x(i+1/2)} = \Phi^{n}_{S_x(i+1/2)} + \Delta t\left[F_{i+1/2,j,k+1/2} - F_{i+1/2,j,k-1/2} - F_{i+1/2,j+1/2,k} + F_{i+1/2,j-1/2,k}\right]$$

$$\Phi^{n+1}_{S_y(j+1/2)} = \Phi^{n}_{S_y(j+1/2)} + \Delta t\left[-F_{i,j+1/2,k+1/2} + F_{i,j+1/2,k-1/2} + F_{i+1/2,j+1/2,k} - F_{i-1/2,j+1/2,k}\right]$$

$$\Phi^{n+1}_{S_z(k+1/2)} = \Phi^{n}_{S_z(k+1/2)} + \Delta t\left[-F_{i+1/2,j,k+1/2} + F_{i-1/2,j,k+1/2} + F_{i,j+1/2,k+1/2} - F_{i,j-1/2,k+1/2}\right]$$

which explicitly preserves $\nabla\cdot\mathbf{B} = 0$ (in difference form) if it is initially true.

The vector potential is considered to be defined at edge centers:

$$\Phi_S = \int_S \mathbf{B}\cdot\mathbf{n}\,dS = \int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \oint_{\partial S} \mathbf{A}\cdot d\,\mathbf{l}$$

Thus, e.g.,

$$(A_x)_{i,j+1/2,k+1/2} \equiv \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} A_x\,dx$$

and

$$(B_x)_{i+1/2,jk} = -\frac{1}{\Delta z_k}\left[(A_y)_{i+1/2,j,k+1/2} - (A_y)_{i+1/2,j,k-1/2}\right] + \frac{1}{\Delta y_j}\left[(A_z)_{i+1/2,j+1/2,k} - (A_z)_{i+1/2,j-1/2,k}\right]$$

We can use this to set up divergence-free initial conditions or boundary conditions.

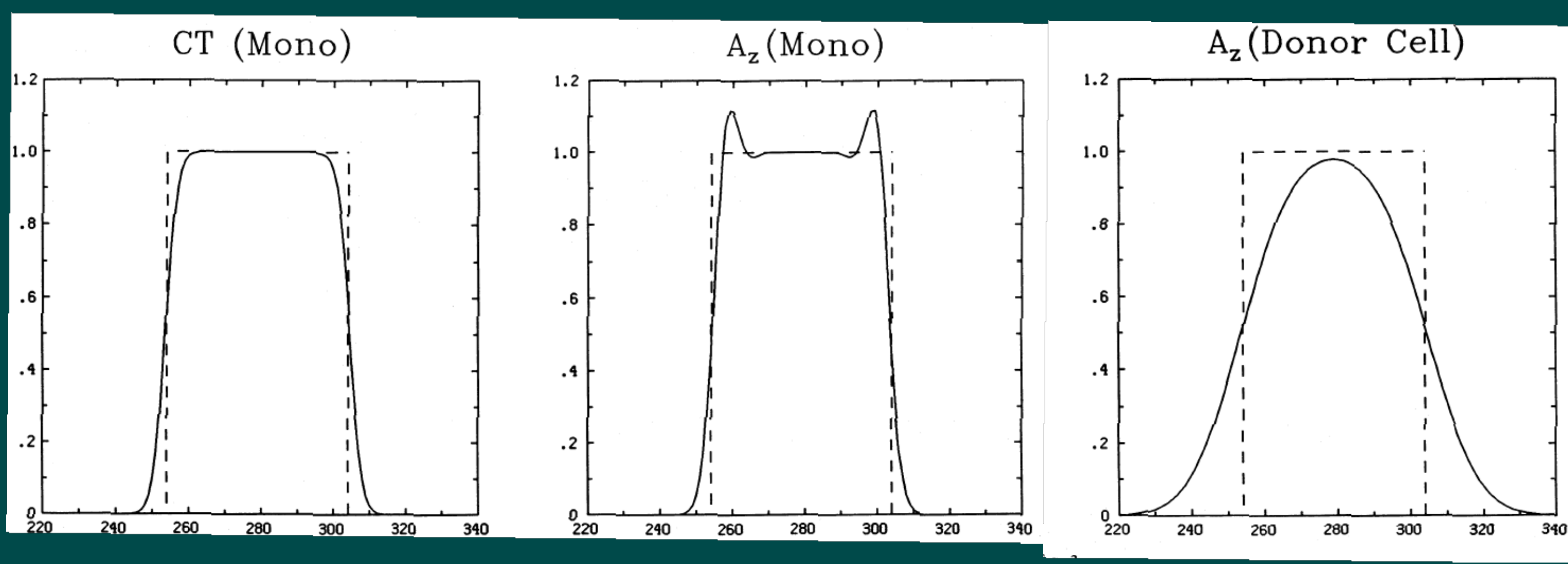
# Constrained transport method – 4

In practice we can factor out the fluxes entirely and work directly with $\mathbf{B}$. The procedure is:

1. Initialize $\mathbf{B}$ on zone faces using the (known) initial vector potential. Initialize $\rho$, $\mathbf{u}$, etc. Staggered grids are used for scalar and vector quantities.

2. Compute $\mathrm{F}$'s on zone edges using upwind differencing of $\mathbf{B}$ and $\mathbf{u}$. A variety of methods can be used (upwind, van Leer, piecewise parabolic, etc.).

3. Write $\Phi$'s in terms of $\mathrm{B}$'s and substitute for $\Phi$ and $\mathrm{F}$ in the flux update equations. This gives us update equations for the magnetic field.




Square-wave advection example from Evans & Hawley (1988) – constrained transport with piecewise linear (van Leer) advection, vector potential differencing, donor cell (first-order upwind).

# Elliptic projection method (Ramshaw 1983)

Also known as *divergence cleaning*. We assume that our update method for $\mathbf{B}$ will produce some nonzero $\operatorname{div}\mathbf{B}$. Every few timesteps (or as appropriate), we project the numerical solution onto the space of divergence-free (*solenoidal*) vector fields.

An arbitrary vector field $\mathbf{B}$ can be written

$$\mathbf{B} = \nabla\phi + \nabla\times\mathbf{A}$$

for some $\phi$ and $\mathbf{A}$. Take the divergence to obtain a Poisson equation for $\phi$:

$$\nabla\cdot\mathbf{B} = \nabla^2\phi$$

So if our update method yields a field $\mathbf{B}^*$ given an initial field $\mathbf{B}^n$, we solve

$$\nabla^2\phi^{n+1} = \nabla\cdot\mathbf{B}^*$$

and take

$$\mathbf{B}^{n+1} = \mathbf{B}^* - \nabla\phi^{n+1}$$

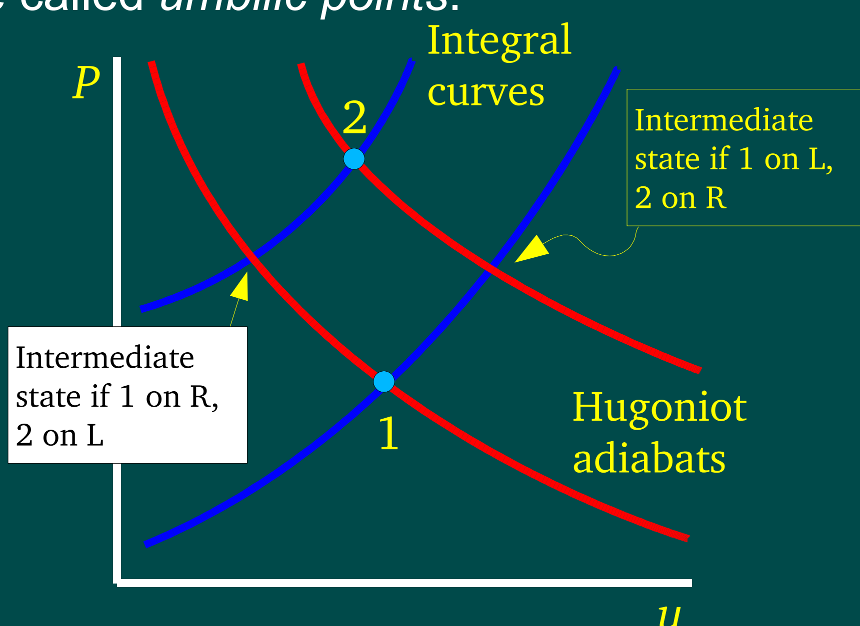
# Godunov methods for MHD

The Riemann problem for MHD is complicated by the MHD characteristic structure.

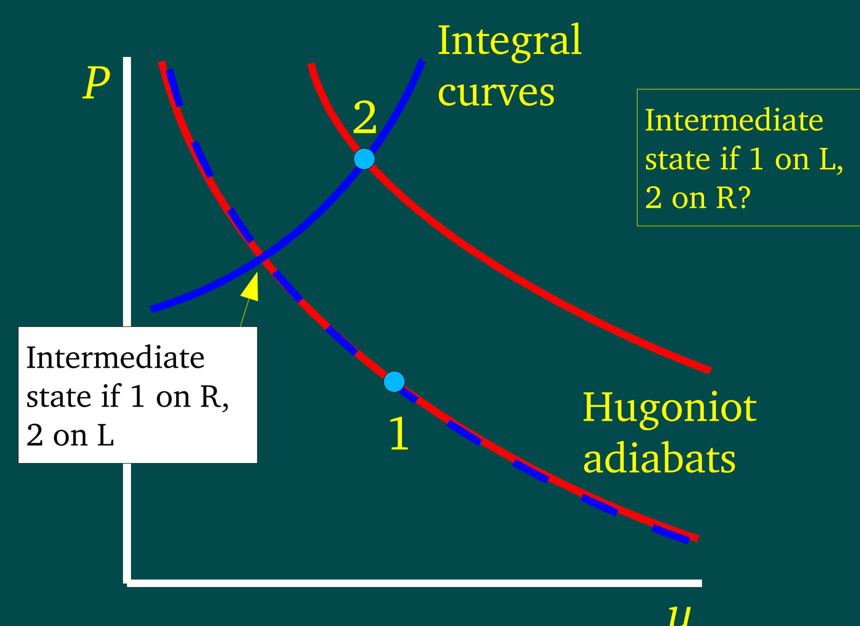
A *hyperbolic system of equations* has a Jacobian matrix with all real-valued eigenvalues. For a *strictly hyperbolic* system of equations the eigenvalues are also unique.

The 1D compressible Euler equations are strictly hyperbolic as long as the equation of state is *convex* (ie. $(a^2 = \partial P/\partial \rho)_s > 0$).

However, for MHD we have three families of characteristics (plus one entropy wave), with the characteristic speeds depending on direction. In some directions two or more speeds are the same, causing strict hyperbolicity to be lost. The points in phase space where this occurs are called *umbilic points*.




Strictly hyperbolic system




Non-strictly hyperbolic system

# Godunov methods – 2

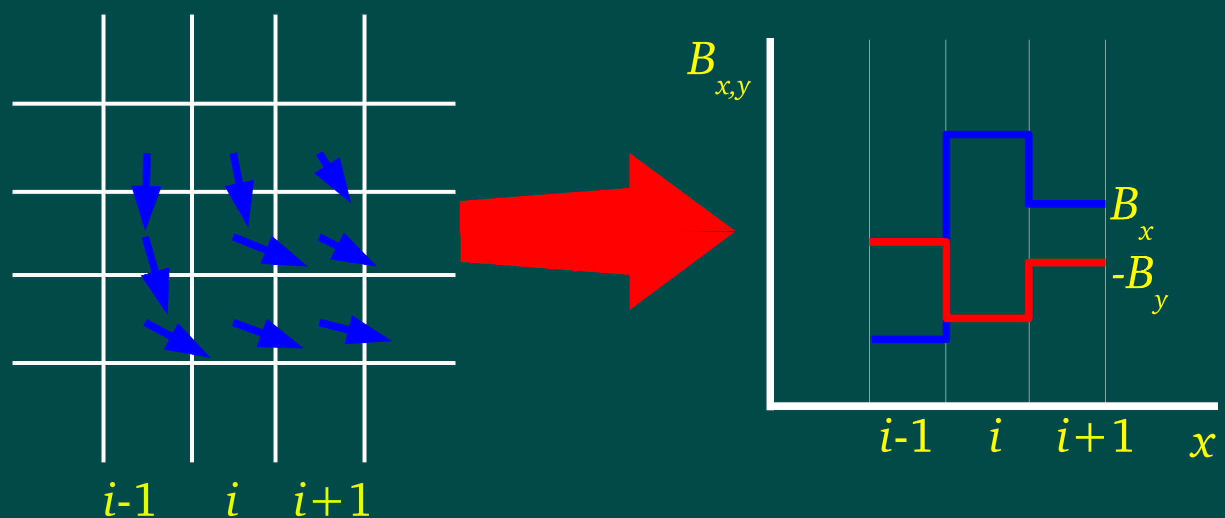
Two more mathematical issues:

- Shocks and rarefactions in the Euler equations are *genuinely nonlinear* – so we can define an *entropy criterion* to pick out physically valid solutions.

  **Example**: *Lax entropy condition*: a wave in the $p$th family with speed $s$ connecting states $q_l$ and $q_r$ is admissible only if $\lambda_p(q_l) > s > \lambda_p(q_r)$.

  However, for MHD, waves can fail to be genuinely nonlinear – what condition should be used then?

- In multidimensional problems, a 1D Riemann solver (used as the basis of an operator-split Godunov method) cannot ignore the other directions. We may see a jump in components of $\mathbf{B}$ across an interface that makes it appear as though $\nabla \cdot \mathbf{B} \neq 0$ on the interface.



# Godunov methods – 3

An example of a 1D Godunov method for MHD is the method by Zachary & Colella (1992) (multidimensional piecewise-linear version by Zachary, Malagoli, & Colella 1994; PPM version by Dai & Woodward 1997).

Assume gradients in sweep direction ($x$) only. Also: in 1D MHD, $B_x$ must be constant in time.

In the conservative basis $\mathbf{U} = [\rho, \rho u_x, \rho u_y, \rho u_z, B_y, B_z, \rho E]^{\mathrm{T}}$, where

$$\rho E = \frac{1}{2}\rho u^2 + \frac{P}{\gamma - 1} + \frac{B_y^2 + B_z^2}{8\pi}$$

the flux vector is

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u_x \\ \rho u_x^2 + P + \frac{B_y^2 + B_z^2}{8\pi} \\ \rho u_x u_y - \frac{B_x B_y}{4\pi} \\ \rho u_x u_z - \frac{B_x B_z}{4\pi} \\ u_x B_y - u_y B_x \\ u_x B_z - u_z B_x \\ \left(\rho E + P + \frac{B_y^2 + B_z^2}{8\pi}\right) u_x - \frac{B_x}{4\pi}(u_y, u_z)\cdot(B_y, B_z) \end{bmatrix}$$

where $\dfrac{\partial \mathbf{U}}{\partial t} + \dfrac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0$

# Godunov methods – 4

Working in the basis $\mathbf{W} = [\tau \equiv 1/\rho, u_x, u_y, u_z, B_y, B_z, P]^{\mathrm{T}}$, we compute the left and right eigenvectors $\mathbf{l}_k$ and $\mathbf{r}_k$ of the Jacobian matrix

$$\mathbf{A} \equiv \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \left[\frac{\partial \mathbf{U}}{\partial \mathbf{W}}\right]^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{W}}$$

There are seven eigenvectors of each type corresponding to the eigenvalues

$$\lambda_0 = u_x$$

$$\lambda_1 = u_x - a_f \quad \lambda_2 = u_x - v_A \quad \lambda_3 = u_x - a_s$$

$$\lambda_4 = u_x + a_s \quad \lambda_5 = u_x + v_A \quad \lambda_6 = u_x + a_f$$

The eigenvectors are chosen to be orthonormal ($\mathbf{l}_i \cdot \mathbf{r}_j = \delta_{ij}$) and are replaced with limiting values when the fast and slow wave speeds are the same. This allows us to define a basis for the subspace spanned by the nondegenerate eigenvectors even when some of the full set are degenerate.

Each interface problem defined by $\mathbf{W}_{\mathrm{L}}$ and $\mathbf{W}_{\mathrm{R}}$ is then decomposed into a linear combination of right eigenvectors:

$$\mathbf{W}_R - \mathbf{W}_L = \sum_{k=1}^{7} \alpha_k \mathbf{R}_k, \qquad \mathbf{R}_k \equiv \mathbf{r}_k\left(\frac{1}{2}(\mathbf{W}_L + \mathbf{W}_R)\right)$$

# Godunov methods – 5

In the absence of a rigorous entropy condition, we cannot determine which direction is the "upwind" direction, and thus which weak solutions are physical.

Zachary & Colella use the entropy wave itself to determine the upwind direction: construct the intermediate states

$$\mathbf{W}_L^* = \mathbf{W}_L + \sum_{\lambda^-} \alpha_k \mathbf{R}_k$$

$$\mathbf{W}_R^* = \mathbf{W}_R - \sum_{\lambda^+} \alpha_k \mathbf{R}_k$$

where the sums are taken over waves moving to the left (–) or right (+) *relative to the entropy wave*. The mean entropy wave speed is defined via

$$\lambda_0^* \equiv \frac{1}{2}\left[\lambda_0(\mathbf{W}_L^*) + \lambda_0(\mathbf{W}_R^*)\right]$$

Depending on the sign of $\lambda_0^*$, we take the upwind state to be $\mathbf{W}_L^*$ or $\mathbf{W}_R^*$. In computing the flux we then add characteristic contributions in this state according to whether each reaches the zone interface during the timestep or not.

This "entropy condition" works well in practice, though it can be used to generate unphysical weak solutions... this is a reason for caution with all current MHD Godunov methods.

# Godunov methods – 6

We can overcome the $\operatorname{div}\mathbf{B}$ issue for the Riemann problem by pretending $\nabla\cdot\mathbf{B}\neq 0$ (Powell 1994). If we allow this the ideal MHD equations look like

$$\frac{\partial\mathbf{B}}{\partial t}+\nabla\times(\mathbf{B}\times\mathbf{u})=\mathbf{u}\nabla\cdot\mathbf{B}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0$$

$$\frac{\partial\rho\mathbf{u}}{\partial t}+\nabla\cdot\left[\rho\mathbf{u}\mathbf{u}+\left(P+\frac{1}{8\pi}B^2\right)\boldsymbol{I}-\frac{1}{4\pi}\mathbf{B}\mathbf{B}\right]=\frac{1}{8\pi}\mathbf{B}\nabla\cdot\mathbf{B}$$

and we add an eighth eigenvalue in 1D. Thus the 1D equations now have eigenvalues (for sweeps in the *x*-direction)

$$\lambda_0=u_x$$

$$\lambda_1=u_x-a_f \quad \lambda_2=u_x-v_A \quad \lambda_3=u_x-a_s$$

$$\lambda_4=u_x+a_s \quad \lambda_5=u_x+v_A \quad \lambda_6=u_x+a_f$$

$$\boxed{\lambda_7=u_x}$$

new eigenvalue

The eighth characteristic field can be interpreted as an advection term for $\nabla\cdot\mathbf{B}$. A Riemann solver based on this set of equations is called an *8-wave Riemann solver*.

# Brio-Wu (1988) MHD shock tube problem

MHD counterpart to Sod shock tube problem:

$$\rho_{\mathrm{L}} = 1 \qquad u_{x,\mathrm{L}} = u_{y,\mathrm{L}} = 0 \qquad P_{\mathrm{L}} = 1 \qquad B_{y,\mathrm{L}} = 1$$

$$\rho_{\mathrm{R}} = 0.125 \qquad u_{x,\mathrm{R}} = u_{y,\mathrm{R}} = 0 \qquad P_{\mathrm{R}} = 0.1 \qquad B_{y,\mathrm{L}} = -1$$

$$B_x = 0.75 \qquad \gamma = 2$$

$t = 0.1$