Nuclear Astrophysics
I. Preliminaries

Karlheinz Langanke

GSI & TU Darmstadt

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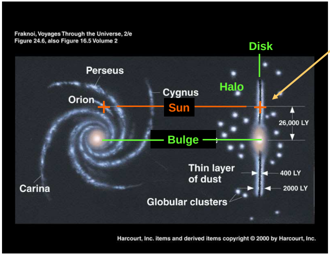
What is nuclear astrophysics?

Nuclear astrophysics aims at understanding the nuclear processes that take place in the universe. These nuclear processes generate energy in stars and contribute to the nucleosynthesis of the elements.

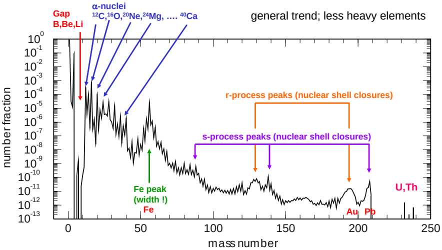
3. The solar abundance distribution



solar abundances:

Elemental (and isotopic) composition of Galaxy at location of solar system at the time of it's formation

Hydrogen mass fraction	X = 0.71
Helium mass fraction	Y = 0.28
Metallicity (mass fraction of everything else)	Z = 0.019
Heavy Elements (beyond Nickel) mass fraction	4E-6



N. Grevesse and A. J. Sauval, Space Science Reviews **85**, 161

Abundance

Nuclear processes conserve number of nucleons:

$$n = \sum_i n_i A_i$$

n number of nucleons per cm^3, $n \approx \frac{\rho}{m_u} = \rho N_A$

n_i number of nuclear species i

Abundance: $Y_i = \frac{n_i}{n} \Rightarrow n_i = \rho N_A Y_i$ (changes in density are factored out)

Mass fraction: $X_i = \frac{n_i m_i}{\rho} = \frac{n_i A_i m_u}{\rho} = Y_i A_i$

From conservation of number of nucleons: $\sum_i Y_i A_i = \sum_i X_i = 1$

Electron Abundance

From charge neutrality:

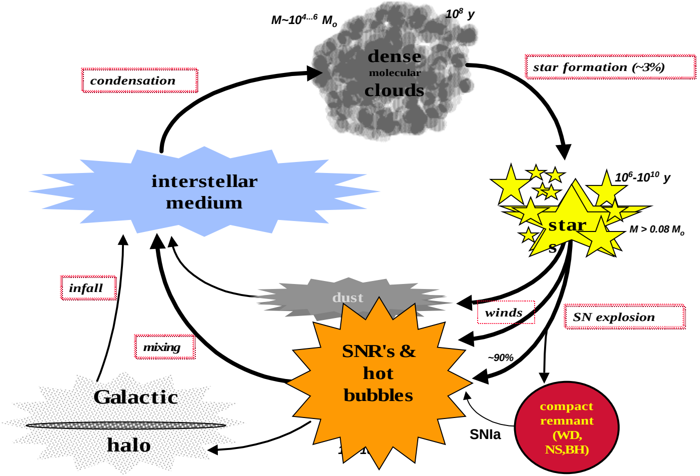
$$n_e = \sum_i n_i Z_i = n \sum_i Y_i Z_i$$

Introducing: $Y_e = \dfrac{n_e}{n}$

$$Y_e = \sum Y_i Z_i$$

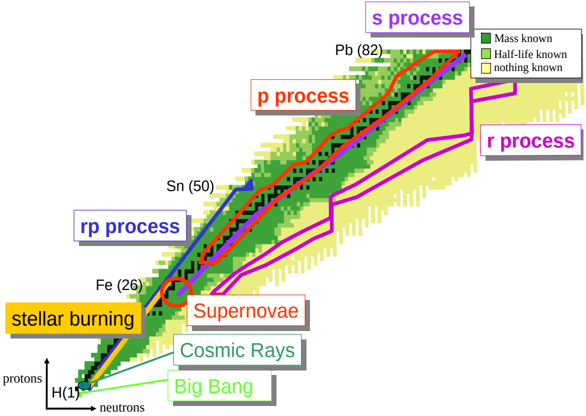
In general one cannot define a lepton abundance. Lepton number is not locally conserved (neutrinos leave the system).

Hoyle's cosmic cycle

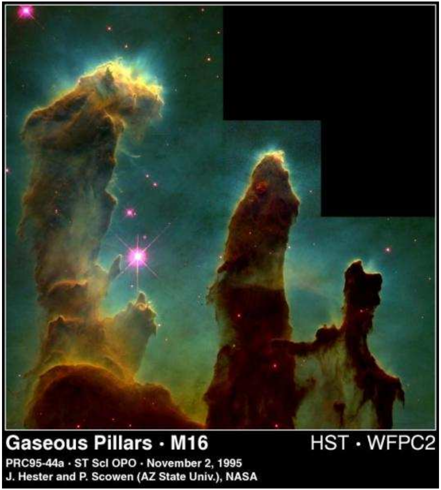


Nucleosynthesis processes

In 1957: Burbidge, Burbidge, Fowler, Hoyle, [Rev. Mod. Phys. **29**, 547 (1957)] suggested the synthesis of the elements in stars.



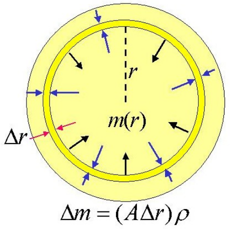
Star formation



- Stars are formed from the contraction of molecular clouds due to their own gravity.
- Contraction increases temperature and eventually nuclear fusion reactions begin. A star is born.
- Contraction time depends on mass: 10 millions years for a star with the mass of the Sun; 100,000 years for a star 11 times the mass of the Sun.

The evolution of a Star is governed by gravity

What is a Star?



in equilibrium: gravity ↔ pressure

- A star is a self-luminous gaseous sphere.
- Stars produce energy by nuclear fusion reactions. A star is a self-regulated nuclear reactor.
- Gravitational collapse is balanced by pressure generated from nuclear reactions:
  $dF_{grav} = -G\frac{m(r)dm}{r^2} = dF_{press} = [(P(r+dr) - P(r))dA$
- Further, equation needed to describe the pressure as function of density, composition (nuclear reactions), temperature (heat transport) → Equation of State (EOS)
- Star evolution, lifetime and death depends on mass. Two groups:
  - Stars with masses less than 8 solar masses (white dwarfs)
  - Stars with masses greater than 8 solar masses (supernova explosions)

Types of processes

Transfer (strong interaction)

$$^{15}\mathrm{N}(p,\alpha)^{12}\mathrm{C}, \qquad \sigma \simeq 0.5 \text{ b at } E = 2.0 \text{ MeV}$$

Capture (electromagnetic interaction)

$$^{3}\mathrm{He}(\alpha,\gamma)^{7}\mathrm{Be}, \qquad \sigma \simeq 10^{-6} \text{ b at } E = 2.0 \text{ MeV}$$

Weak (weak interaction)

$$p(p,e^{+}\nu)d, \qquad \sigma \simeq 10^{-20} \text{ b at } E = 2.0 \text{ MeV}$$

Reaction rates

Basic ingredient to determine the change in composition are the reaction rates:
Suppose a nuclear reaction:

$$1 + 2 \rightleftarrows 3 + 4$$

$$\frac{dN_1}{dt} = r_{34} - r_{12}$$

r number of reactions per cubic centimeter and per second.
In these lectures we will discuss how to determine the different rates. Let us consider first the equilibrium situation: $\frac{dN_i}{dt} = 0$
This occurs at very high temperatures ($T \gtrsim 5$ GK = 430 keV) when the reactions proceed much faster than the dynamical evolution of the system (Big Bang, supernovae,...).
In equilibrium it suffices to know the chemical potentials

$$\mu_1 + \mu_2 = \mu_3 + \mu_4$$

Nuclear Statistical Equilibrium

Processes mediated by the strong and electromagnetic interaction are in equilibrium. Normally neutrinos can escape and weak equilibrium cannot be achieved.
Processes of creation and destruction are in equilibrium

$$(Z, A) \rightleftarrows Zp + Nn + \gamma' s$$

Composition determined by (T, ρ, Y_e). Entropy ($\sim T^3/\rho$) is the main parameter determining the abundances. High entropies (low ρ, high T) favor free nucleons. Small entropies (high ρ, low T) favor bound nuclei.

Nuclear Statistical Equilibrium

During stellar burning, NSE is achieved at temperatures in access of about $(3-4)10^9$ K ($\sim 250-350$ keV). At such temperatures reactions via the strong and electromagnetic interaction proceed in both directions as temperature is high enough

- to overcome Coulomb barrier
- to dissociate nuclei by photons from the high-energy tail of the Planck distribution

Nuclear Statistical Equilibrium

Consequences:

- All nuclei in the reaction network are connected to each other
- Complete chemical equilibrium
- There exist 2 'outside' constraints:
  1. Mass conservation
  2. Charge neutrality (Y_e is fixed by environment)

The 2 conserved quantities imply 2 independent chemical potentials, which are chosen as μ_p (protons) and μ_n (neutrons).

Nuclear abundances in NSE

NSE implies:

$$\mu(Z,A) = Z\mu_p + (A-Z)\mu_N$$

with the chemical potentials given by (Boltzmann)

$$\mu(Z,A) = m(Z,A)c^2 + kT \ln\left[\frac{n(Z,A)}{G(Z,A)}\left(\frac{2\pi\hbar^2}{m(Z,A)kT}\right)^{3/2}\right]$$

and the partition function:

$$G(Z,A) = \sum_i (2J_i+1)e^{-E_i/kT}$$

Proton, neutron: G=2 (two spins!)

Solution to NSE

Saha equation:

$$Y(Z,A) = \frac{G(Z,A)A^{3/2}}{2^A}(\rho N_A)^{A-1} Y_p^Z Y_n^N \left(\frac{2\pi\hbar^2}{m_u kT}\right)^{3/2(A-1)} e^{E_b(Z,A)/kT}$$

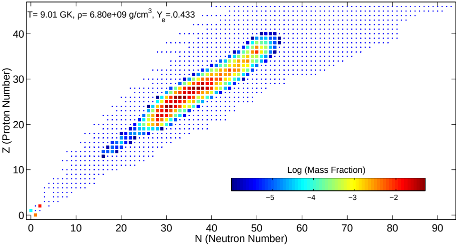
with $E_b(Z,A) = (Nm_n + Zm_p - M(Z,A))c^2$
and the constraints

- $\sum_i Y_i A_i = 1$ (conservation number nucleons)
- $\sum_i Y_i Z_i = Y_e$ (charge neutrality)

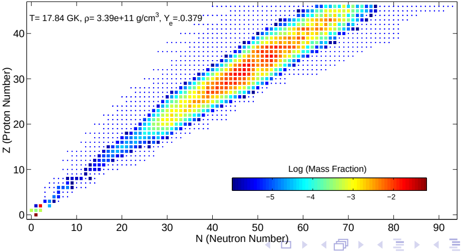
- High density: favors large A ($\sim \rho^{(A-1)}$)
- High temperature: favors light nuclei ($\sim T^{-3/2(A-1)}$)

NSE: Nuclear composition during core-collapse supernova.

presupernova stage



neutrino trapping



Non-equilibrium: Reaction network

In astrophysical environments many reactions occur simultaneously. This is called a reaction network. Nuclei can be produced and destroyed by several reactions.
This leads to equations of the type:

$$\frac{dN_a}{dt} = -\sum_b \lambda_b N_a + \sum_b \lambda_b N_b - \sum_{b,c,d} N_a N_b \langle \sigma v \rangle_{c,d} + \sum_{b,c,d} N_c N_d \langle \sigma v \rangle_{a,b}$$

or

$$\frac{dY_a}{dt} = -\sum_b \lambda_b Y_a + \sum_b \lambda_b Y_b - N_A \rho \sum_{b,c,d} Y_a Y_b \langle \sigma v \rangle_{c,d} + N_A \rho \sum_{b,c,d} Y_c Y_d \langle \sigma v \rangle_{a,b}$$

Transmission through barrier step

Consider a particle coming from $-\infty$ and the barrier:
$V(x) = 0 \text{ for} x < 0$ and $V(x) = -V_0 \text{ for} x > 0$.

The solutions are plane waves with $k_1 = \frac{\sqrt{2mE}}{\hbar}$ for $x < 0$ and $k_2 = \frac{\sqrt{2m(E+V_0)}}{\hbar}$ for $x > 0$.

$x < 0$: $\phi(x) = A_1 \exp\{ik_1 x\} + B_1 \exp\{-ik_1 x\}$
$x > 0$: $\phi(x) = A_2 \exp\{ik_2 x\}$

At $x = 0$ the matching conditions for the wave functions and their derivatives yield:
$A_1 + B_1 = A_2;\ A_1 - B_1 = \frac{k_2}{k_1} A_2$.

This yields $A_2 = A_1 \frac{2k_1}{k_1+k_2}$

Transmission coefficient

The transmission coefficient defines the ratio of the transmitted flux to the incoming flux.

$$T = \frac{j_{\text{out}}}{j_{\text{in}}} = \frac{k_{\text{out}}|\phi_{\text{out}}|^2}{k_{\text{in}}|\phi_{\text{in}}|^2}$$

with $j = \frac{\hbar k}{m}|\phi|^2$.
For the example follows:

$$T = \frac{k_2|A_2|^2}{k_1|A_1|^2} = \frac{k_2 4k_1^2}{(k_1+k_2)^2 k_1} = \frac{4k_1k_2}{(k_1+k_2)^2}$$

Transmission through constant barrier

Consider a barrier which is $V(x) = +V_0$ for $x_1 < x < x_2 = x_1 + d$.
The particle again comes from $-\infty$ with energy $E < V_0$.
The transmission coefficient can be calculated as:

$$T = [1 + \frac{V_0^2}{V_0^2 - (2E - V_0)^2} \sinh^2(k_2 d)]^{-1}$$

with $k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$.

In the case $k_2 d >> 1$ the sinh function reduces to $\sinh^2(k_2 d) \approx \frac{1}{4} \exp\{2k_2 d\}$.

Thus, $T \sim \exp\{-\frac{2}{\hbar}\sqrt{2m(V_0 - E)}d\}$.

Coulomb potential

General potential V(x): numerical solution.

Often WKB approximation is sufficient. Example is the spherical Coulomb potential $V(r) = \frac{Z_1 Z_2 e^2}{r}$ where one finds

$$T \approx \exp\{-\frac{2}{\hbar}\int_{r_1}^{r_2} \sqrt{2m(V(r)-E)}dr\} = \exp\{-2\pi\eta\}$$

with the Sommerfeld parameter $\eta = \sqrt{\frac{\mu}{2E}}\frac{Z_1 Z_2 e^2}{\hbar}$.
r_1, r_2 are the classical turning points.

Total reaction cross section

The total reaction cross section is given by

$$\sigma = \frac{\pi}{k_{\rm in}^2} \sum_l (2l+1) T_l$$

In astrophysical applications one is often interested in cross sections at rather low energies. Then often only $l = 0$ partial waves (s-waves) contribute noticeably to the cross section as the centrifugal barrier reduces the transmission in other partial waves (Note, however, resonances!).

Total reaction cross section

1. step barrier (neutrons)
   $k_1 = \frac{\sqrt{2\mu E}}{\hbar}$; $k_2 = \frac{\sqrt{2\mu(E+Q)}}{\hbar}$ with $E << Q$
   Then k_2 roughly constant, $k_1 << k_2$ and $T_{l=0} = \frac{4k_1k_2}{(k_1+k_2)^2} \approx 4\frac{k_1}{k_2}$.
   For the cross section follows $\sigma = \frac{\pi}{k_1^2} 4\frac{k_1}{k_2} \sim \frac{1}{k_1}$.
   Indeed, $\sigma \sim \frac{1}{\sqrt{E}} \sim \frac{1}{v}$ is a good approximation for low-energy neutrons.
2. constant barrier (charged particles)
   $\sigma = \frac{\pi}{k_{\rm in}^2}\exp\{-2\pi\eta\} = \frac{\hbar^2\pi}{2\mu E}\exp\{-2\pi\eta(E)\}$

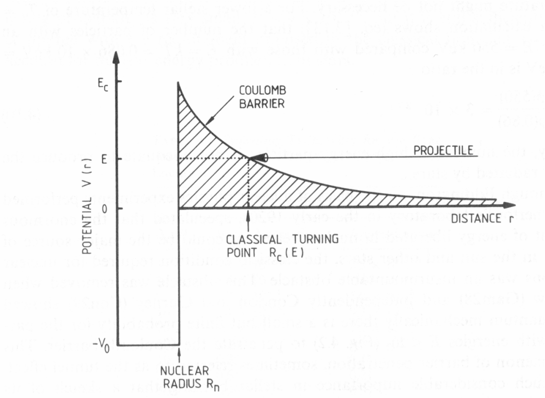
   Penetration through Coulomb barrier at low energies is extremely energy-dependent. Some of this energy dependence is known and can be factorized from the cross section. This leads to the defnition of the

   astrophysical S-factor $S(E) = \sigma(E) E \exp\{2\pi\eta(E)\}$.

   For non-resonant reactions, $S(E)$ is a slowly varying function in E.

Charged-particle cross section

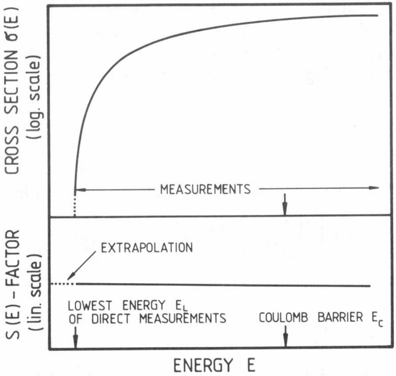
Stars' interior is a plasma made of charged particles (nuclei, electron). Nuclear reactions proceed by tunnel effect. For $p + p$ reaction Coulomb barrier 550 keV, but the typical energy in the sun is only 1.35 keV.

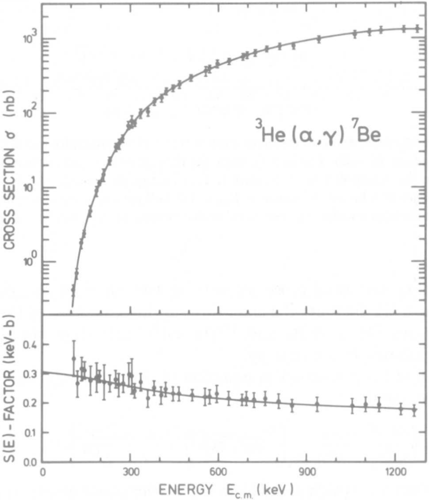


cross section: $\sigma(E) = \frac{1}{E} S(E) e^{-2\pi\eta}$; $\quad \eta = \frac{Z_1 Z_2 e^2}{\hbar}\sqrt{\frac{\mu}{2E}} = \frac{b}{E^{1/2}}$

Astrophysical S factor

S factor allows accurate extrapolations to low energy.





Stellar reaction rate

Consider N_a and N_b particles per cubic centimeter of particle types a and b. The rate of nuclear reactions is given by:

$$r = N_a N_b \sigma(v) v$$

In stellar environment the velocity (energy) of particles follows a thermal distribution that depends on the type of particles.

- Nuclei (Maxwell-Boltzmann): $\phi(v) = N 4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right)$

The product σv has to be averaged over the velocity distribution $\phi(v)$

$$\langle \sigma v \rangle = \int_0^\infty \int_0^\infty \phi(v_a)\phi(v_b)\sigma(v) v dv_a dv_b$$

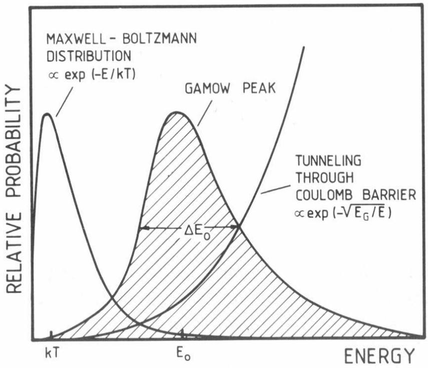
Changing to center-of-mass coordinates, integrating over the cm-velocity and using $E = \mu v^2/2$

$$\langle \sigma v \rangle = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma(E) E \exp\left(-\frac{E}{kT}\right) dE$$

Gamow window

Using definition of S factor:

$$\langle \sigma v \rangle = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) \exp\left[-\frac{E}{kT} - \frac{b}{E^{1/2}}\right] dE$$



Gamow window

Assuming that S factor is constant over the Gamow window and approximating the integrand by a Gaussian one gets:

$$\langle \sigma v \rangle = \left(\frac{2}{\mu}\right)^{1/2} \frac{\Delta}{(kT)^{3/2}} S(E_0) \exp\left(-\frac{3E_0}{kT}\right)$$

$$\begin{aligned} E_0 &= 1.22[\mathrm{keV}](Z_1^2 Z_2^2 \mu T_6^2)^{1/3} \\ \Delta &= 0.749[\mathrm{keV}](Z_1^2 Z_2^2 \mu T_6^5)^{1/6} \end{aligned}$$

(T_x measures the temperature in 10^x K.)
Examples for solar conditions:

reaction	E_0 [keV]	$\Delta/2$ [keV]	$I_{\max}$	T dependence of $\langle \sigma v \rangle$
p+p	5.9	3.2	1.1×10^{-6}	$T^{3.9}$
p+^{14}N	26.5	6.8	1.8×10^{-27}	T^{20}
α+^{12}C	56.0	9.8	3.0×10^{-57}	T^{42}
^{16}O+^{16}O	237.0	20.2	6.2×10^{-239}	T^{182}

It depends very sensitively on temperature!