Simple Concepts and Estimates… (continuation)

Width of a narrow resonance

open closed

$$H(t) = H_0 + V(t) \quad (|V| << |H_0|)$$ time-dependent Hamiltonian

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$ …expansion of ψ in the basis of H_o

$$\hat{H}_0\phi_n = E_n\phi_n \Rightarrow \psi = \sum_n c_n(t)\phi_n e^{-iE_n t/\hbar}$$

$$i\hbar\frac{dc_k}{dt} = \sum_n c_n(t)\langle\phi_k|V|\phi_n\rangle e^{i\omega_{kn}t}, \quad \omega_{kn} = (E_k - E_n)/\hbar$$

As initial conditions, let us assume that at t=0 the system is in the state $\phi_{0.}$ That is,

$$c_n(0) = \begin{cases} 1 \text{ for n} = 0 \\ 0 \text{ for n} \neq 0 \end{cases}$$

If the perturbation is weak, in the first order, we obtain:

$$i\hbar\frac{dc_k}{dt} = \langle\phi_k|V|\phi_0\rangle e^{i\omega_{k0}t}$$

Furthermore, if the time variation of *V* is slow compared with $\exp(i\omega_{ko}t)$, we may treat the matrix element of V as a constant. In this approximation:

$$c_k(t) = \frac{\langle \phi_k | V | \phi_0 \rangle}{E_k - E_0}\left(1 - e^{i\omega_{k0}t}\right)$$

The probability for finding the system in state k at time t if it started from state 0 at time t=0 is:

$$|c_k(t)|^2 = 2\frac{|\langle \phi_k | V | \phi_0 \rangle|^2}{(E_k - E_0)^2}\left(1 - \cos\omega_{k0}t\right)$$

The total probability to decay to a group of states within some interval labeled by *f* equals:

$$\sum_{k \in f} |c_k(t)|^2 = \frac{2}{\hbar^2}\int \frac{|\langle \phi_k | V | \phi_0 \rangle|^2}{\omega_{k0}^2}\left(1 - \cos\omega_{k0}t\right)\rho(E_k)\,dE_k$$

The transition probability per unit time is

$$\mathcal{W} = \frac{d}{dt} \sum_{k \in f} \left| c_k(t) \right|^2 = \frac{2}{\hbar^2} \int \left| \langle \phi_k | V | \phi_0 \rangle \right|^2 \frac{\sin \omega_{k0} t}{\omega_{k0}} \rho(E_k) dE_k$$

Since the function sin(x)/x oscillates very quickly except for x~0, only small region around E_0 can contribute to this integral. In this small energy region we may regard the matrix element and the state density to be constant. This finally gives:

$$\mathcal{W}_{0 \to f} = \frac{2\pi}{\hbar} \left| \langle \phi_f | V | \phi_0 \rangle \right|^2 \rho(E_f)$$

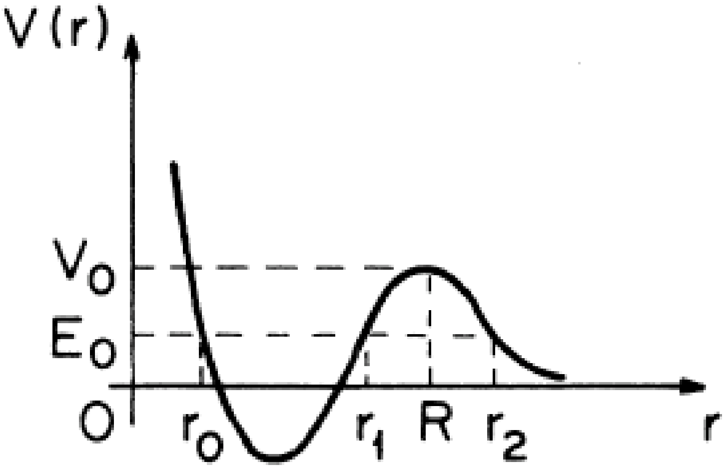
Fermi's golden rule

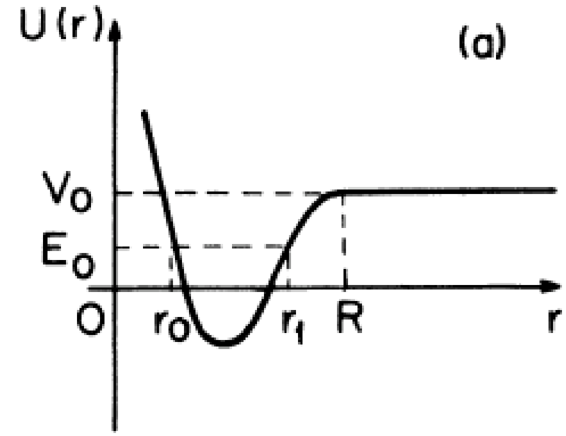
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

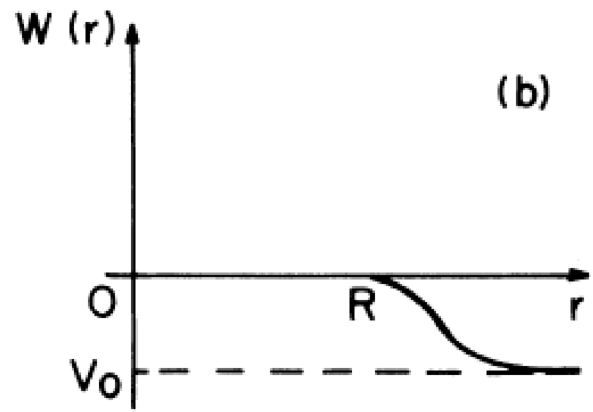
Two potential approach to tunneling

(decay width and shift of an isolated quasistationary state)

A. Gurvitz, Phys. Rev. A 38, 1747 (1988); A. Gurvitz et al., Phys. Rev. A69, 042705 (2004)



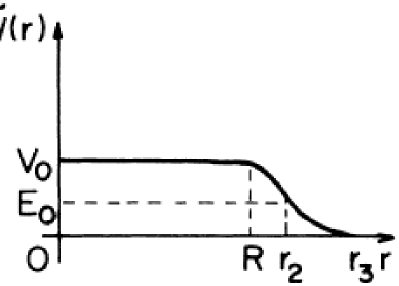




$$V(r) = U(r) + W(r)$$

open closed scattering

$$\tilde{W} = W + V_0$$



$$\Gamma = \frac{4m}{\hbar^2 k} \left| \int_R^\infty \phi_0(r) W(r) \chi_k(r) dr \right|^2$$

eigenstate of T+U with E=E_0

eigenstate of T+$\tilde{W}$ with E=E_0

Isolated Breit-Wigner resonance; Fermi's golden rule

Final simple expressions for the width and energy shift:

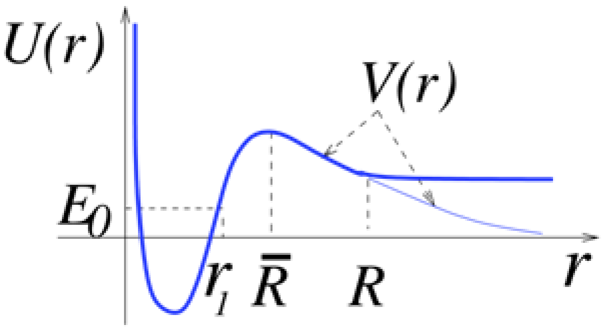
$$E = E_0 + \Delta - \frac{i}{2}\Gamma$$

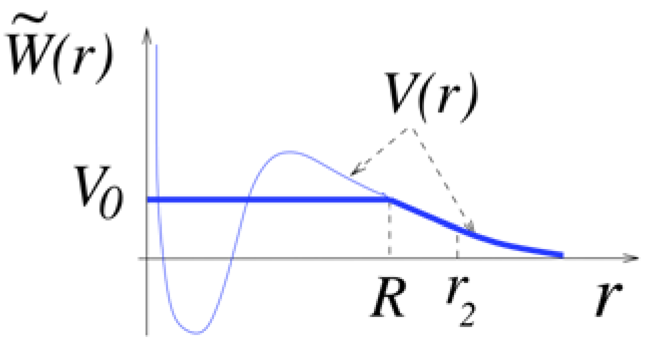
$$\Gamma = \frac{8\hbar(V_0 - E_0)}{k} \left|\phi_0(R)\chi_k(R)\right|^2$$

$$\Delta = -\frac{\hbar^2\alpha}{mk} \left|\phi_0(R)\right|^2 \left[2\alpha\chi_k(R) Re\chi_k^{(+)}(R) - k\right]$$

$$\alpha^2 = 2m(V_0 - E_0)/\hbar$$

Improvement:



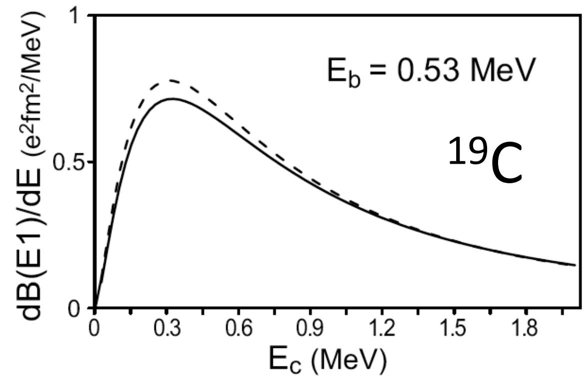


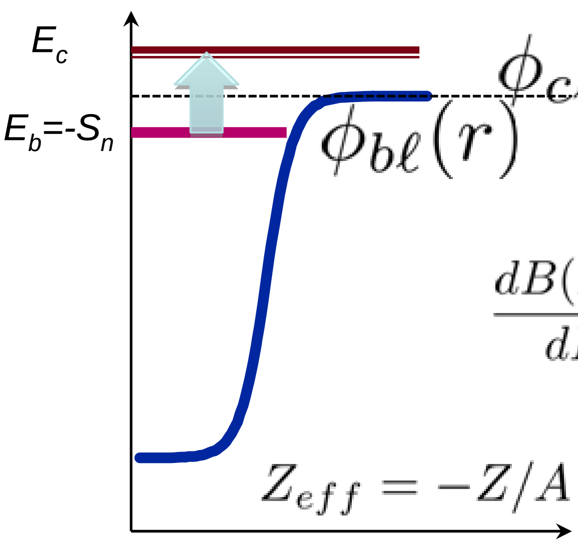
$\bar{r}-\bar{R}$ (fm)	η	Γ_{MTPA} (MeV)	$\frac{(\Delta\Gamma)}{\Gamma}$
$0h_{11/2}$ Gamow state: E_{res}=1.5 MeV, Γ =4.918 E-18 MeV			
0.17	0.11	4.665 E-18	5%
1.55	0.02	4.87 E-18	1%
3.09	0.003	4.909 E-18	0.2%
$2s_{1/2}$ Gamow state: E_{res}=1.5 MeV, Γ =6.695 E-14 MeV			
0.26	0.06	6.577 E-14	2%
1.66	0.01	6.675 E-14	0.3%
3.24	0.001	6.692 E-14	0
$0i_{13/2}$ Gamow state: E_{res}=1 MeV, Γ =1.834 E-6 MeV			
0.18	0.15	1.736 E-6	5%
1.45	.04	1.814 E-6	1%
2.99	.007	1.831 E-6	0.1%
$1f_{5/2}$ Gamow state: E_{res}=1 MeV, Γ =9.271 E-2 MeV			
0.18	0.13	8.998 E-2	3%
1.38	0.035	8.856 E-2	4%
2.96	0.005	8.373 E-2	10%

Low-lying dipole strength for weakly bound systems

Two-body case, 1-neutron halo

Bertulani and Baur, Nucl. Phys. A480, 615 (1988)
Nagarajan, Lenzi, Vitturi, Eur. Phys. J. A24, 63 (2005)





$$\frac{dB(E1)}{dE} = \frac{3}{4\pi}(Z_{eff}e)^2\langle \ell 010|\ell' 0\rangle^2 \left| \int dr \phi^*_{b\ell}(r)\phi_{c\ell'}(E_c, r) r^3 \right|^2$$

$$\phi_{b\ell}(r) = N_b h_\ell^{(1)}(i\alpha r) \qquad \phi_{c\ell'}(r) = \sqrt{\frac{2\mu k}{\hbar^2\pi}} j_{\ell'}(kr)$$

$$\alpha^2 = -2\mu E_b/\hbar^2 \qquad k^2 = 2\mu E_c/\hbar^2$$

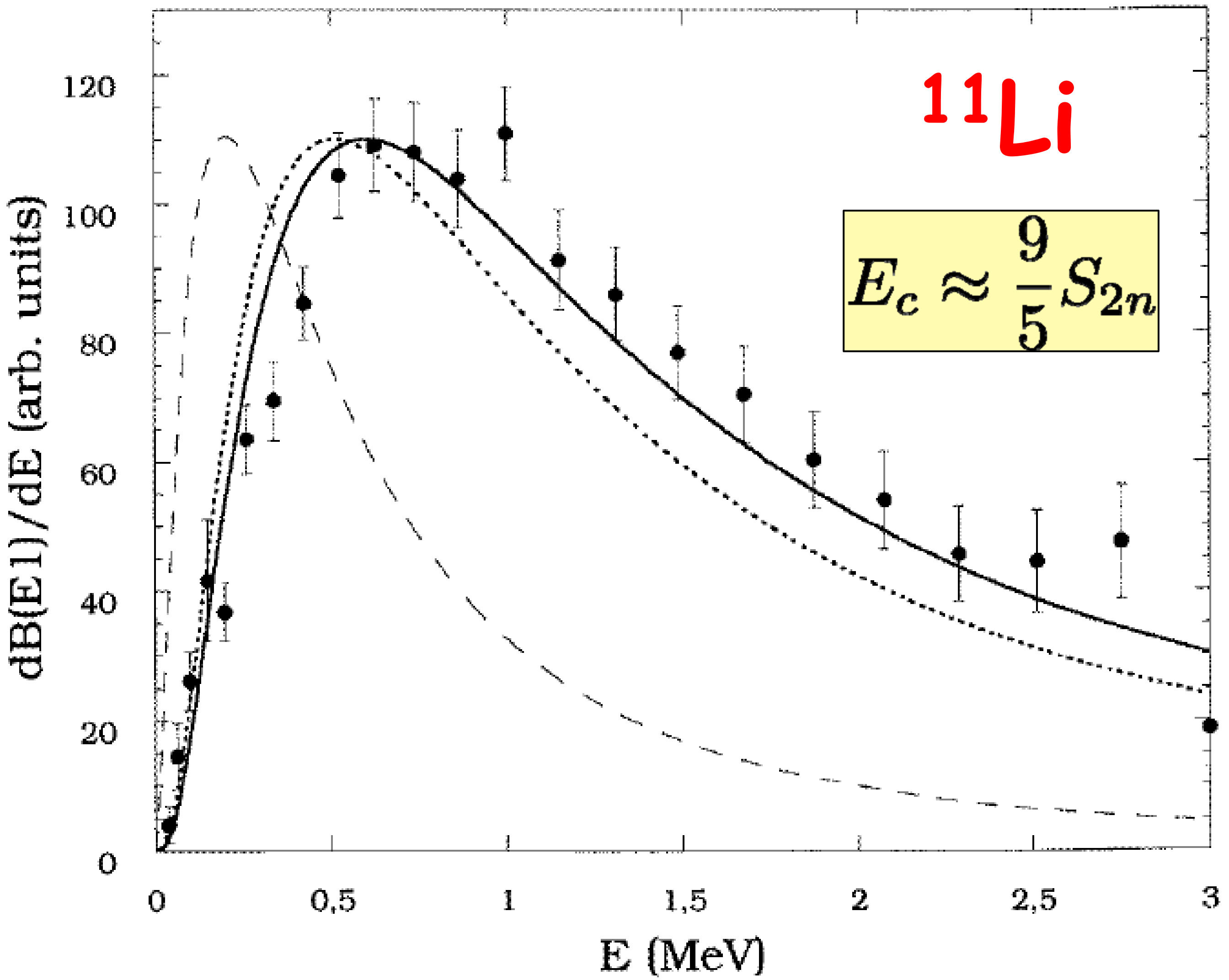
For $\ell = 0 \rightarrow \ell' = 1$ transition:

$$\frac{dB(E1)}{dE} \propto \frac{\sqrt{S_n} E_c^{3/2}}{(E_c + S_n)^4}$$

It has maximum at $E_c = \frac{3}{5} S_n$

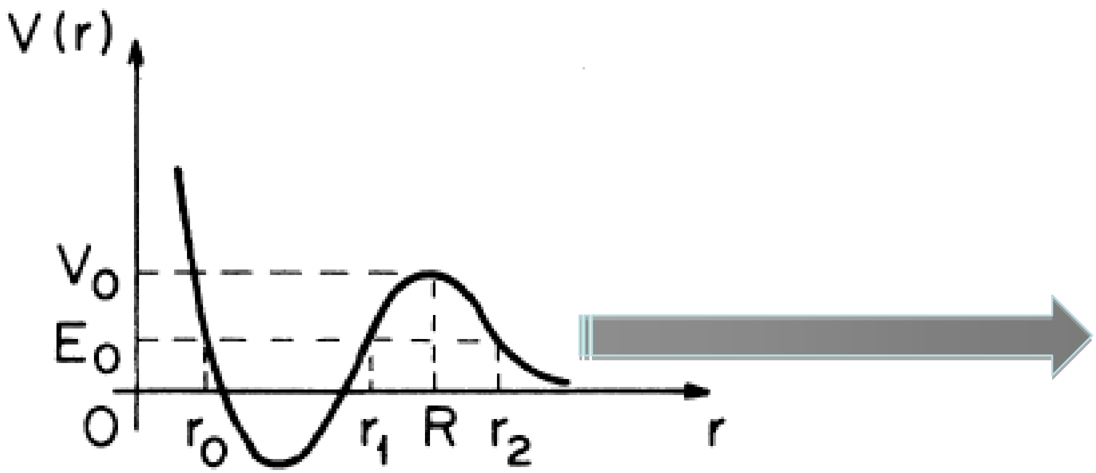
Three-body case, 2-neutron halo

Pushkin, Jonson, and Zhukov, J. Phys. G 22, L95 (1996



Quasistationary states and Gamow states

Quasistationary States



For the description of a decay, we demand that far from the force center there be only the outgoing wave. The macroscopic equation of decay is

$$\frac{dN}{dt} = -wN; \quad N = N_0 e^{-wt}$$

N is a number of radioactive nuclei, i.e., number of particles inside of sphere *r=R:*

$$N \sim \int |\psi^2| d^3 r$$

We should thus seek a solution of the form

$$\psi = \psi(r)e^{-iE_0 t/\hbar - wt/2} = \psi(r)e^{-iEt/\hbar}$$

$$E = E_0 - i\frac{\Gamma}{2}; \quad \Gamma = \hbar w$$

J.J. Thompson, 1884
G. Gamow, 1928

relation between decay width and decay probability

The time dependent equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\Delta + V(r)\right]\psi$$

can be reduced by the above substitution to the stationary equation

$$\left[E + \frac{\hbar^2}{2m}\Delta - V(r)\right]\psi(r) = 0$$

The boundary condition

$$\lim_{r\to\infty}\frac{d\ln\psi(r)}{dr} = i\frac{\sqrt{2mE}}{\hbar}$$

takes care of the discrete complex values of *E*

Since the energy E is complex, the momentum k is also complex. Asymptotically

$$\psi(r) \sim e^{ikr}/r$$

$$k = \frac{\sqrt{2mE_0}}{\hbar} - i\frac{\Gamma}{4\hbar}\sqrt{\frac{2m}{E_0}}$$

Therefore

$$\psi(r) \sim e^{i\frac{\sqrt{2mE_0}}{\hbar}r} \cdot \frac{e^{\frac{\Gamma}{4\hbar}\sqrt{\frac{2m}{E_0}}r}}{r}$$

Looks scary but nothing to worry about!

What is a physical interpretation of the asymptotic growth of the wave function at large r? At any time t_0 we find at a given distance from the center those particles which were emitted at a previous time

$$t = t_0 - r/v$$

However, on the account of the exponential time dependence, the amplitude of the wave function at the center at the earlier time was greater than it is at t_0. Indeed

$$\frac{\Gamma}{4\hbar}\sqrt{\frac{2m}{E_0}}r = \frac{w}{2}\frac{r}{v} = \frac{w}{2}(t_0 - t)$$

$$\Rightarrow \quad \psi(r, t_0) = \psi(0, t_0 - r/v)$$

(the exponential temporal decrease of the wave function amplitude is complemented by its exponential spatial increase, and the divergence of the resonance wave function assures that the particle number is conserved)

Outgoing flux and width of the Gamow state

Humblet and Rosenfeld: Nucl. Phys. 26, 529 (1961)

$$\left[\hat{T} + \hat{V}\right]\Psi = \left(e - i\frac{\Gamma}{2}\right)\Psi$$

$$\left[\hat{T} + \hat{V}\right]\Psi^* = \left(e + i\frac{\Gamma}{2}\right)\Psi^*$$

$$\hbar \oint_S \vec{j}\, d\vec{S} = \Gamma \int_V \rho\, d^3 r$$

$$\vec{j} = \frac{\hbar}{2mi}\left(\Psi^* \vec{\nabla}\Psi - \Psi \vec{\nabla}\Psi^*\right) \qquad \rho = \Psi^*\Psi \qquad \vec{\nabla}\vec{j} - \frac{\Gamma}{\hbar}\rho = 0 \qquad \left(\vec{\nabla}\vec{j} + \frac{\partial \rho}{\partial t} = 0\right)$$

S can be taken as a sphere of radius *R*:

$$\Gamma = \frac{\hbar R^2 \int j_r\, d\Omega}{\int_{V_R} \rho\, d^3 r}$$

An extremely useful expression!

When can we talk about "existence" of an unbound system?

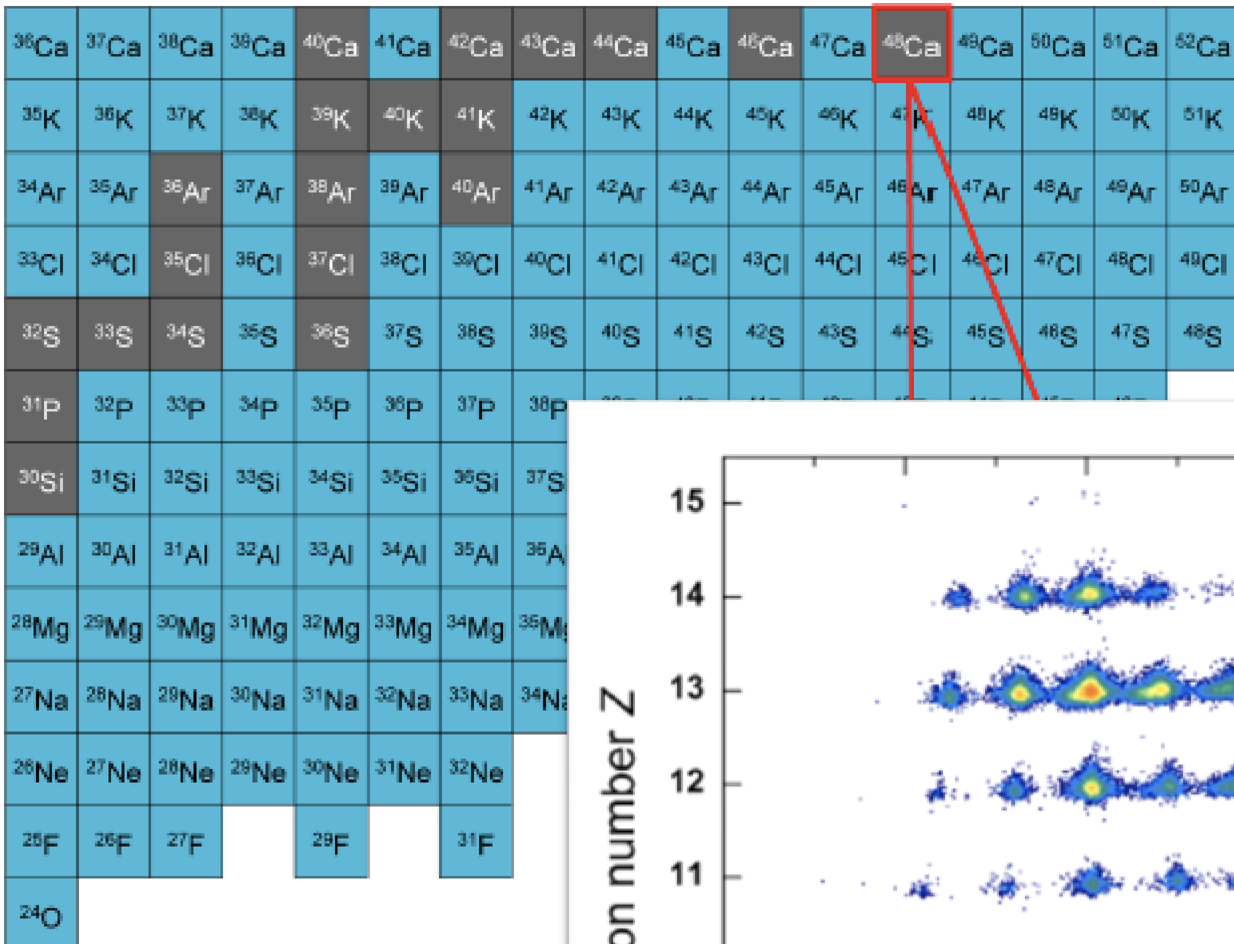
$$T_{1/2} = \ln 2 \frac{\hbar}{\Gamma}, \qquad \hbar = 6.58 \cdot 10^{-22} \text{ MeV} \cdot \text{sec}$$

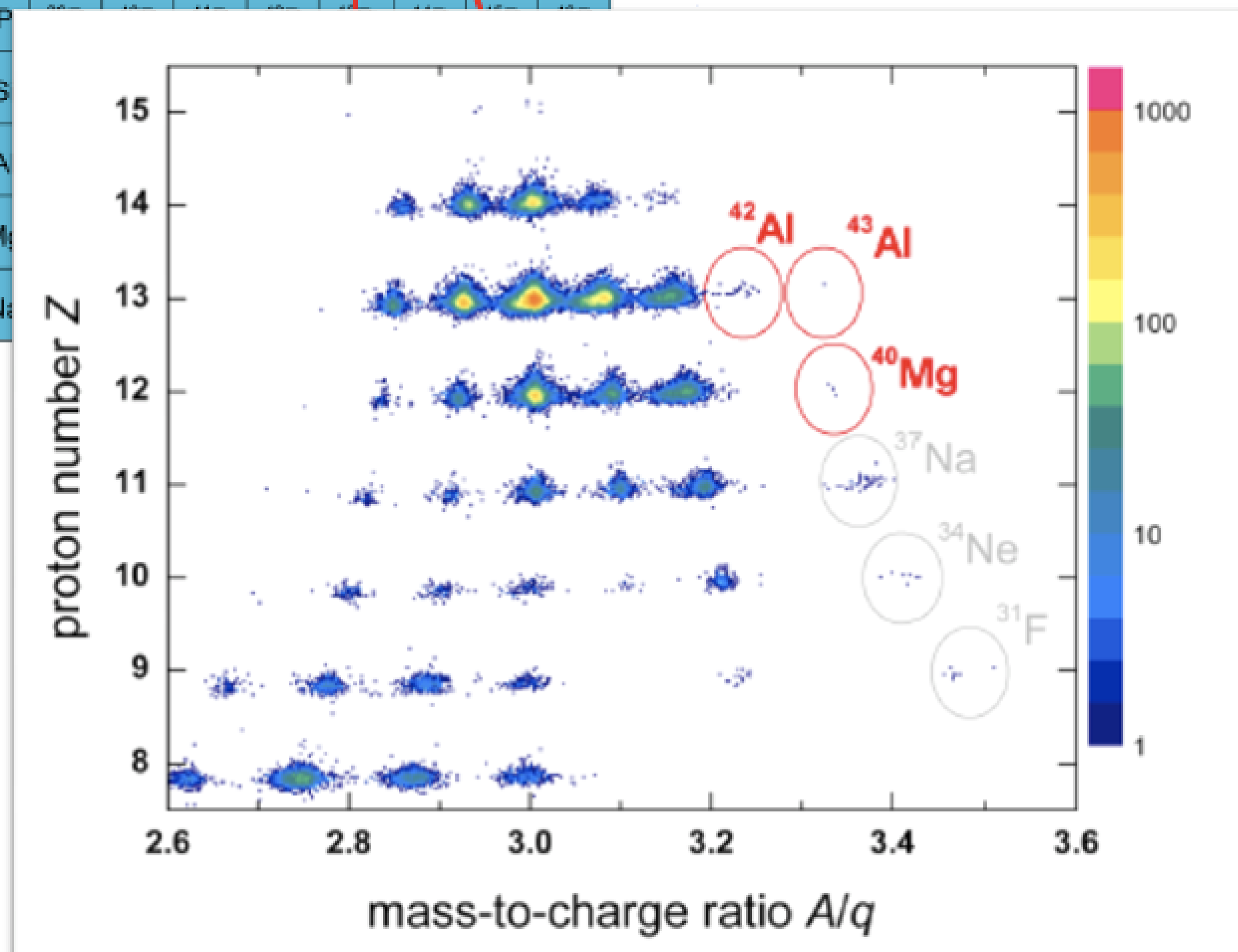
$$T_{s.p.} \approx 3 \cdot 10^{-22} \,\text{sec} = 3 \,\text{baby sec}.$$

A typical time associated with the s.p. motion

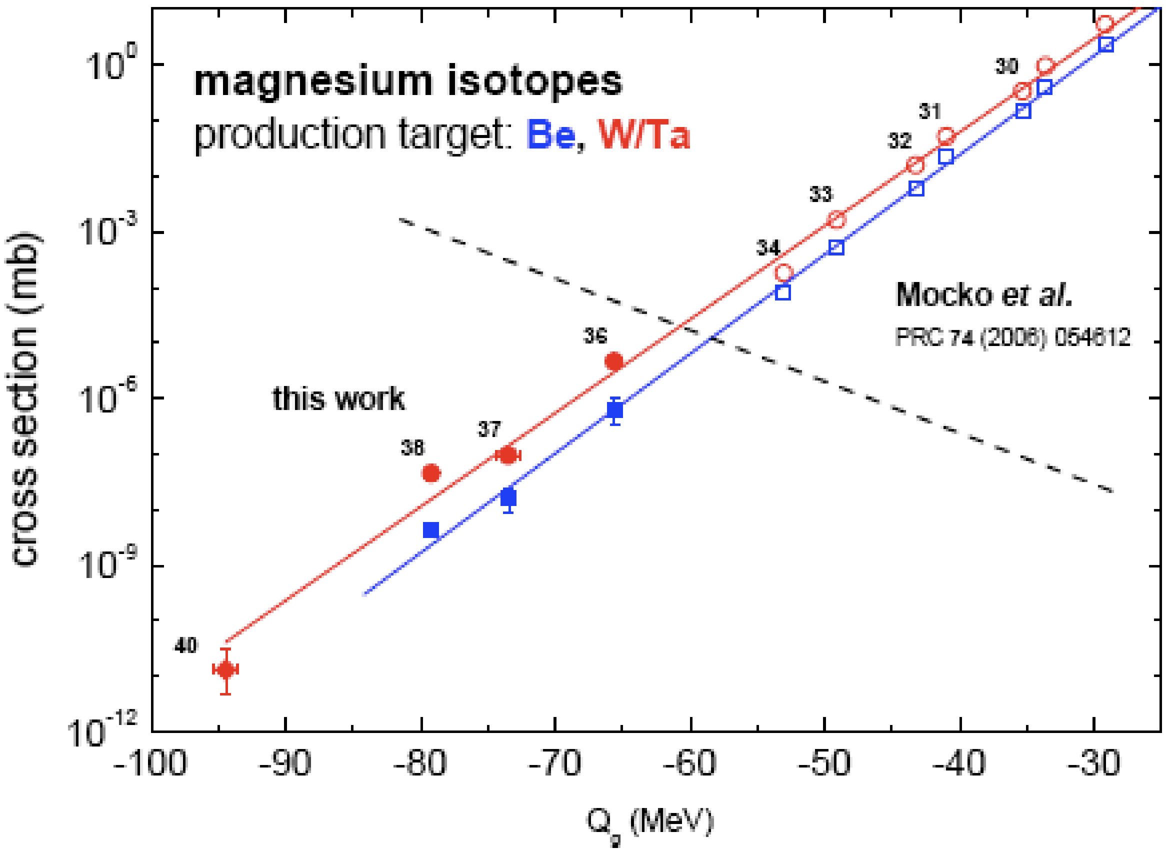
$$T_{1/2} >> T_{s.p.}$$

$$\Rightarrow \quad \Gamma << 1 MeV$$



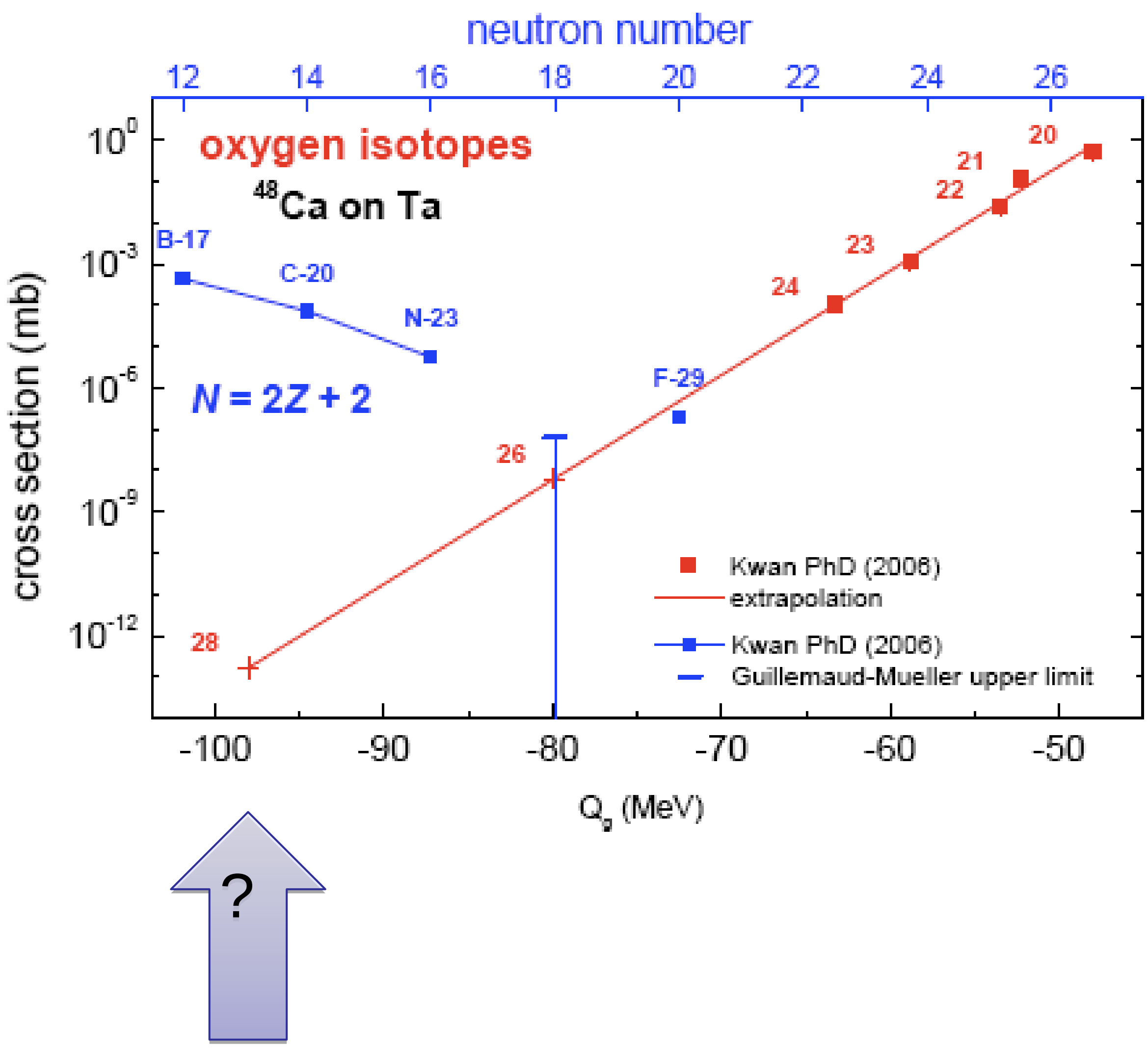


Baumann et al., ENAM'08



Extrapolates very well!

the difference of mass excess of projectile and target


neutron number
12
14
16
18
20
22
24
26
oxygen isotopes
48Ca on Ta
B-17
C-20
N-23
N = 2Z + 2
F-29
20
21
22
23
24
26
28
Kwan PhD (2006)
extrapolation
Kwan PhD (2006)
Guillemaud-Mueller upper limit
cross section (mb)
10^0
10^-3
10^-6
10^-9
10^-12
-100
-90
-80
-70
-60
-50
Qg (MeV)
?